Generalized eigenfunctions and a Borel Theorem on the Sierpinski Gasket.[*]

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1 Introduction

There is a well developed theory (see [5, 9]) of analysis on certain types of fractal sets, of which the Sierpinski Gasket (SG) is the simplest non-trivial example. In this theory the fractals are viewed as limits of graphs, and notions analogous to the Dirichlet energy and the Laplacian are constructed as renormalized limits of the corresponding objects on the approximating graphs. The nature of this construction has naturally led to extensive study of the eigenfunctions of this Laplacian, and to functional-analytic notions based on the eigenfunctions. However, more recent work [7, 2] has examined other elementary functions on SG, including analogues of polynomials, analytic functions and certain exponentials. A forthcoming paper [8] will extend this investigation to study smooth bump functions and a method for partitioning smooth functions subordinate to an open cover.

In the present work we prove there are exponentially decaying generalized eigenfunctions on a blow-up of SG with boundary (which we denote SG_∞), proving:

Theorem 1.1. *For each $\lambda < 0$ and $j \in \mathbb{N}$ there is a smooth function E_λ^j on SG_∞ such that for each j we have $(\Delta + \lambda)E_\lambda^j = -jE_\lambda^{j-1}$. Moreover E_λ^j decays exponentially away from the boundary point of SG_∞ and satisfies $|E_\lambda^j| \leq j!|\lambda|^{-j}$.*

There are sufficiently many of these generalized eigenfunctions that they can be used to prove a Borel-type theorem on SG_∞, thereby answering a question asked in [7, 2]. Using the term *jet* for the sequence of values of the natural derivatives at a junction point of SG our result may be summarized as:

Theorem 1.2. *Given an arbitrary jet there is a smooth function on SG_∞ with that jet at the boundary point.*

Our motivation for studying generalized eigenfunctions and for proving Theorem 1.1 was to prove Theorem 1.2. The structure of the paper reflects this motivation: apart

from some background in Section 2 our first results (in Section 3) are those showing that Theorem 1.2 follows from Theorem 1.1 and some known results about localized eigenfunctions on SG. Section 4 is then devoted to the construction of the generalized eigenfunctions and the proof of Theorem 1.1.

Acknowledgments

2 Setting

We give a brief description of some parts of the theory of analysis on the Sierpinski Gasket, more details of which are in [9]. For the general theory of analysis on fractals the standard reference is [5].

SG and *SG*$_\infty$

The Sierpinski gasket SG is the unique non-empty compact set in $\mathbb{R}^2$ that is invariant under the iterated function system $f_i = \frac{1}{2}(x + q_i)$, $i = 0, 1, 2$ in the sense that $SG = \bigcup_{i=0}^{2} f_i(SG)$, where the points q_i are the vertices of an equilateral triangle. For $m \in \mathbb{N}$ and $(i_1, i_2, \ldots, i_m) \in \{0, 1, 2\}^m$ we call $f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_m}(SG)$ a *cell of level m*. The points $V_0 = \{q_j : j = 0, 1, 2\}$ are the boundary of SG and we view SG as the limit of graphs Γ_m with vertices defined inductively by $V_m = \bigcup_{i=0}^{2} f_i(V_{m-1})$ and edge relation $x \sim_m y$ if x and y are in the same m-cell. The set of all vertices is $V_\infty = \bigcup_m V_m$ and the *junction points* are $V_\infty \setminus V_0$. We let μ be the usual self-similar probability measure on SG with $\mu(f_w(SG)) = 3^{-|w|}$, and also use μ to denote the obvious extension to SG_∞.

The infinite blow-up of SG with boundary point q_0 is denoted SG_∞ and defined by

$$SG_\infty = \bigcup_{m=0}^{\infty} f_0^{\circ(-n)}(SG) \tag{2.1}$$

This is the simplest blow-up of SG; we could also consider arbitrary sequences of blow-ups $\bigcup_{m=1}^{\infty} f_{i_1}^{-1} f_{i_2}^{-1} \cdots f_{i_m}^{-1}(SG)$, but all that have a boundary point necessarily have $\{i_m\}$ to be constant after some m_0 and are isometric, so among those with boundary it suffices to consider SG_∞ (see Lemma 2.3 in [11]). The work in this paper will crucially use that we are on a blow-up with boundary. We refer to [10, 11] for more information about blow-ups and the Laplacian on SG_∞.

Laplacian, derivatives and jets

Each graph approximation Γ_m of SG supports a graph Laplacian Δ_m defined at non-boundary points $x \in V_m \setminus V_0$, and we define a Laplacian Δ at junction points of SG as

a renormalized limit of the graph Laplacians

$$\Delta_m u(x) = \sum_{y \sim_m x} (u(y) - u(x)) \tag{2.2}$$

$$\Delta u(x) = \frac{3}{2} \lim_{m \to \infty} 5^m \Delta_m u(x). \tag{2.3}$$

A continuous function u is in the domain of the Laplacian, $u \in \text{dom}(\Delta)$, if there is a continuous f such that the right side of (2.3) converges uniformly to f on $V_\infty \setminus V_0$. Then we write $\Delta u = f$, extending Δu to all points of SG by continuity. The factor $3/2$ in (2.3) is for consistency with an alternative definition of the Laplacian using a renormalized Dirichlet energy (see [5, 9]). We make the obvious definition of $\Delta^k u$ and $\text{dom}(\Delta^k)$, and call a function smooth if it is in $\text{dom}(\Delta^\infty) = \cap_k \text{dom}(\Delta^k)$. One additional property of the Laplacian that we will use extensively is its scaling; it is immediate from (2.3) that for $u \in \text{dom}(\Delta)$

$$\Delta(u \circ f_0^{-1}) = 5(\Delta u) \circ f_0^{-1} \quad \text{on } f_0(SG) \tag{2.4}$$

In addition to the Laplacian there are two derivatives at boundary points, the normal derivative ∂_n and tangential derivative ∂_T, defined by

$$\partial_n u(q_i) = \lim_{m \to \infty} \left(\frac{5}{3}\right)^m (2u(q_i) - u(f_j^{\circ m}(q_{i+1})) - u(f_j^{\circ m}(q_{i+2}))) \tag{2.5}$$

$$\partial_T u(q_i) = \lim_{m \to \infty} 5^m (u(f_j^{\circ m}(q_{j+1})) - u(f_j^{\circ m}(q_{j+2}))) \tag{2.6}$$

(with $q_{i+3} = q_i$). The former exists for any $u \in \text{dom}(\Delta)$ and the latter exists under the additional assumption that Δu is Hölder continuous. Both may be localized to boundary points of cells. The normal derivatives are much better understood than the tangential derivatives, and have considerable application; for this paper their important feature is the *matching condition* for the normal derivatives: if $u \in \text{dom}(\Delta)$ then at any junction point of two cells the normal derivatives corresponding to these cells sum to zero. Conversely, if u is continuous and $\Delta u = f$ on each m-cell then $\Delta u = f$ on SG if and only if f is continuous and the matching condition holds at each point of $V_m \setminus V_0$.

At a boundary point q we will call the values of $\Delta^k u(q)$, $\partial_n \Delta^k u(q)$ and $\partial_T \Delta^k u(q)$ the Laplacian powers, normal derivatives and tangential derivatives, respectively. The j-jet of u at q is $(u(q), \partial_n u(q), \partial_T u(q), \ldots, \Delta^j u(q), \partial_n \Delta^j u(q), \partial_T \Delta^j u(q))$ and the infinite jet is the corresponding sequence of Laplacian powers and derivatives.

Spectral decimation

A useful feature of the Laplacian on SG and SG_∞ is that restricting a Laplcaian eigen-function E_λ to the graph approximation Γ_m produces an eigenfunction of the graph Laplacian, with a shift in the eigenvalue. This phenomenon is known as spectral deci-mation [4, 3, 11]. Specifically, if $(\Delta + \lambda)E_\lambda = 0$ on SG or SG_∞ then $(\Delta_m + \lambda_m)E_\lambda = 0$

on Γ_m, where

$$\lambda_{m-1} = \lambda_m(5 - \lambda_m) \tag{2.7}$$

$$\lambda = \frac{3}{2} \lim_{m \to \infty} 5^m \lambda_m \tag{2.8}$$

$$E_\lambda(y_i) = \frac{2E_\lambda(x_i) + (4 - \lambda_m)(E_\lambda(x_{i+1}) + E_\lambda(x_{i+2}))}{(2 - \lambda_m)(5 - \lambda_m)} \tag{2.9}$$

in which the points x_i are the vertices of an $(m-1)$-cell, and each y_i is the point from V_m opposite to x_i as shown in Figure 1. For proofs we refer to [3] or [9]; an explanation of spectral decimation in a more general context is in [6].
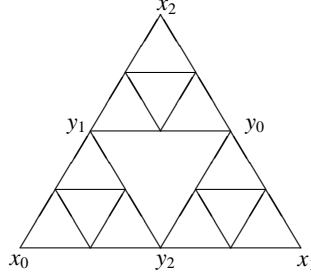

Figure 1: Points x_i in V_{m-1} and y_i in V_m.

We will need one technical result about spectral decimation on SG that is well-known but for which there does not appear to be a proof in the literature.

Lemma 2.1. *For $\lambda < 0$ there is an entire function Ψ such that $\lambda_m = \Psi(5^{-m}\lambda)$.*

Proof. Considering (2.7) we define functions by $2\phi_\pm(\zeta) = 5 \pm \sqrt{25 - 4\zeta}$, so that λ_m is one of $\phi_\pm(\lambda_{m-1})$. Observe that if $\lambda_{m-1} \geq 0$ then $\lambda_m \geq 0$, so from (2.8) and $\lambda < 0$ we must have $\lambda_m < 0$ and $\lambda_m = \phi_-(\lambda_{m-1})$ for all m. The renormalized limit $\Phi(\zeta) = \frac{3}{2} \lim_{m \to \infty} 5^m \phi_-^m(\zeta)$ is analytic in a neighborhood of the origin and has $\Phi'(0) = 3/2$ by virtue of the fact that $\phi_-(\zeta) = \zeta/5 + O(\zeta^2)$ for sufficiently small ζ. It follows that Φ has an analytic inverse $\Psi(\zeta) = \Phi^{-1}(\zeta) = \sum_{k=0}^\infty \alpha_k \zeta^k$ in a neighborhood of 0.

Using (2.7) and (2.8) we find $\Psi(5^{-m}\lambda) = \lambda_m$ for all sufficiently small $5^{-m}\lambda$. This gives a recursion for the coefficients α_k, beginning with $\alpha_0 = 0$, $\alpha_1 = 2/3$ and continuing according to

$$(5^k - 5)\alpha_k = -\sum_{l=1}^{k-1} \alpha_l \alpha_{k-l} \tag{2.10}$$

for $k \geq 2$. An almost identical recursion appears for a different purpose in [7] (as Equation 2.9 and in Theorem 2.7) and their argument shows that $|\alpha_k| \leq C(k!)^{-\log 5/\log 2}$. It follows immediately that Ψ is entire, while $\lambda_m = \Psi(5^{-m}\lambda)$ is true by construction. $\square$

3 The Borel Theorem

In this section we prove Theorem 1.2 under the assumption of Theorem 1.1. First we construct smooth functions with finitely many prescribed values of the Laplacian powers and tangential derivatives at q_0 using known results about the existence of localized eigenfunctions. Then we use Theorem 1.1 and linear algebra to prove that there are smooth functions with finitely many prescribed normal derivatives. Finally we state a precise version of Theorem 1.2 and show that its validity for finite jets gives the full result by a scaling and convergence argument.

Localized eigenfunctions

A curious feature of many highly symmetric fractals is that their Laplacians have localized eigenfunctions. We will not need the details of the theory, for which we refer to [1, 5], but only the existence of two specific eigenfunctions u_1 and u_2 on SG. The values of u_1 on V_1 and u_2 on V_2 are shown in Figure 2. From the values shown we can

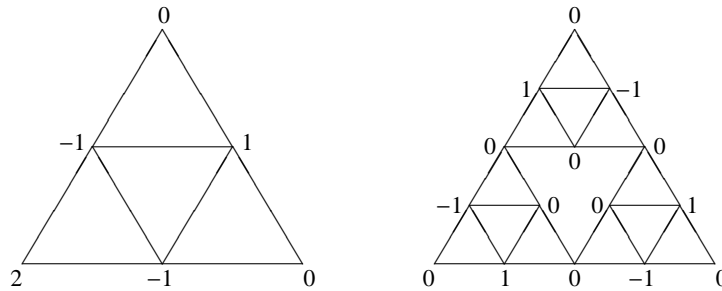Figure 2: The functions u_1 and u_2.

compute u_1 and u_2 at any scale by the method of spectral decimation given in (2.7)-(2.9) (with the caveat that for u_1 the positive root must be taken at the first step of the recursion (2.7)). What is important here is that the normal derivatives of both u_i vanish at the points q_1 and q_2, which we see from (2.5) and the antisymmetry of the u_i on the cells $f_1(SG)$ and $f_2(SG)$. Since the u_i are eigenfunctions we then find that all of the values $\Delta^k u_i$ and $\partial_n \Delta^k u_i$ vanish at q_1 and q_2, and (2.4) shows the same is true for $u_i \circ f_0^{-m}$ at $f_0^m(q_1)$ and $f_0^m(q_2)$ for any m. It follows from the matching condition that

$$u_{i,m} = \begin{cases} u_i \circ f_0^{-m} & \text{on } f_0^{\circ m}(SG) \\ 0 & \text{otherwise} \end{cases}$$

are smooth functions, and therefore are Laplacian eigenfunctions with eigenvalues $-5^m \lambda_i$. For obvious reasons they are called *localized eigenfunctions*.

The jets of the $u_{i,m}$ at q_0 are easily computed. The eigenfunction equations $\Delta u_i = -\lambda_i u_i$ give the higher order terms from the initial ones, so by simple algebra from (2.9)

5

and some symmetry arguments

$$\Delta^k u_{i,m}(q_0) = \begin{cases} 2(-\lambda_1)^k 5^{km} & i = 1 \\ 0 & i = 2 \end{cases}$$

$$\partial_T \Delta^k u_{i,m}(q_0) = \begin{cases} 0 & i = 1 \\ 2(-\lambda_2)^k 5^{km} & i = 2 \end{cases}$$

$$\partial_n \Delta^k u_{i,m}(q_0) = 0 \quad i = 1, 2.$$

With these functions as building blocks we show that there is a smooth function with finitely prescribed values of the Laplacian powers and tangential derivatives at q_0, and whose normal derivatives are all zero.

Lemma 3.1. *For $n \in \mathbb{N}$ and values $\zeta_0, \ldots, \zeta_n$ and $\theta_0, \ldots, \theta_n$ there is $u \in \mathrm{dom}(\Delta^\infty)$ such that $\Delta^k u(q_0) = \zeta_k$, $\partial_n \Delta^k u(q_0) = 0$, and $\partial_T \Delta^k u(q_0) = \theta_k$ for all $0 \le k \le n$.*

Proof. We observe that the vectors

$$(u_{1,m}(q_0), \Delta u_{1,m}(q_0), \ldots, \Delta^n u_{1,m}(q_0)) = 2(1, (-\lambda_1)5^m, \ldots, (-\lambda_1)^n 5^{nm})$$

are linearly independent with respect to m. A similar result is true for the vector of tangential derivatives of $u_{2,m}$. We may then obtain the desired u as a linear combination of the functions $u_{i,m}$ for $0 \le m \le n$ by linear algebra. $\qquad\square$

We remark that this method cannot be applied to prescribe values of the normal derivatives at q_0 using localized eigenfunctions. The structure of the localized eigenfunctions is well understood (see [9]) and non-zero normal derivatives can occur only in "closed loops" circling the holes in the gasket. As each junction point corresponds to a hole of precisely one size, our scaling arguments are not applicable. Similar arguments are needed in Lemma 3.4 below, so any proof of Theorem 1.2 using only localized eigenfunctions would need to be quite different from ours.

Generalized eigenfunctions and normal derivatives

The generalized eigenfunctions produced in Theorem 1.1 are a sufficiently rich class that we can use a finite linear combination of them to match finitely many prescribed normal derivatives at q_0.

Lemma 3.2. *For $n \in \mathbb{N}$, $\lambda < 0$ and values $\eta_0, \ldots, \eta_n$ there is $u \in \mathrm{dom}(\Delta^\infty)$ which is a finite linear combination of the E_λ^j, $0 \le j \le n$, and has $\partial_n \Delta^k u(q_0) = \eta_k$ for all $0 \le k \le n$.*

Proof. Let $a_{j,k} = \partial_n \Delta^k E_\lambda^j(q_0)$. It clearly suffices to show that the matrix $[a_{j,k}]_{j,k=0}^n$ is invertible, so we examine its determinant. Writing the generalized eigenfunction equation $(\Delta + \lambda)E_\lambda^j = -jE_\lambda^{j-1}$ in terms of $a_{j,k}$ we have $a_{j,k} + \lambda a_{j,k-1} = -ja_{j-1,k-1}$, which suggests a column operation on $[a_{j,k}]$. For all columns $k \ge 1$ we replace $a_{j,k}$ with $-ja_{j-1,k-1}$, which makes the first row zero except in the first place simply because

6

$j = 0$ on this row. For concreteness the result of this computation for the determinant in the case $n = 2$ is given below.

$$\begin{vmatrix} a_{0,0} & a_{0,1} & a_{0,2} \\ a_{1,0} & a_{1,1} & a_{1,2} \\ a_{2,0} & a_{2,1} & a_{2,2} \end{vmatrix} = \begin{vmatrix} a_{0,0} & 0 & 0 \\ a_{1,0} & -a_{0,0} & -a_{0,1} \\ a_{2,0} & -2a_{1,0} & -2a_{1,1} \end{vmatrix} = a_{0,0} \begin{vmatrix} -a_{0,0} & -a_{0,1} \\ -2a_{1,0} & -2a_{1,1} \end{vmatrix}$$

This operation can be repeated inductively, because it shows

$$\det\left[a_{j,k} \right]_0^n = a_{0,0} \det\left[-ja_{j-1,k-1} \right]_1^n = a_{0,0} \det\left[-(j+1)a_{j,k} \right]_0^{n-1}$$

and the only change to the matrix at each stage is to multiply each row by a constant and reduce the degree, so the same column operations apply each time. We conclude that $\det[a_{j,k}]_0^n = (-1)^n n! a_{0,0}^{n+1}$, and is non-zero by (4.2) below. $\qquad\square$

Corollary 3.3. *Given values $(\zeta_k, \eta_k, \theta_k)$ for $0 \le k \le n$ there is a finite linear combination u of localized eigenfunctions and generalized eigenfunctions such that $\Delta^k u(q_0) = \zeta_k$, $\partial_n \Delta^k u(q_0) = \eta_k$ and $\partial_T \Delta^k u(q_0) = \theta_k$ for all $0 \le k \le n$.*

Proof. Apply Lemma 3.2 to match the normal derivatives and then Lemma 3.1 to correct the Laplacian powers and tangential derivatives without affecting the values of the normal derivatives. $\qquad\square$

Proof of the Borel theorem

Corollary 3.3 supplies the natural building blocks for obtaining a smooth function with any given jet at q_0. Define for each j functions F_j from which we will determine the Laplacian powers, G_j for the normal derivatives and H_j for the tangential derivatives by requiring that for all $0 \le k \le j$

$$\Delta^k F_j(q_0) = \delta_{j,k} \qquad\qquad \partial_n \Delta^k F_j(q_0) = 0 \qquad\qquad \partial_T \Delta^k F_j(q_0) = 0$$
$$\Delta^k G_j(q_0) = 0 \qquad\qquad \partial_n \Delta^k G_j(q_0) = \delta_{j,k} \qquad\qquad \partial_T \Delta^k G_j(q_0) = 0$$
$$\Delta^k H_j(q_0) = 0 \qquad\qquad \partial_n \Delta^k H_j(q_0) = 0 \qquad\qquad \partial_T \Delta^k H_j(q_0) = \delta_{j,k}$$

where $\delta_{j,k}$ is the Kronecker delta. The natural goal is to construct a smooth function with prescribed (infinite) jet by using the terms of the jet as coefficients in a series with functions like the F_j, G_j and H_j. To make the series converge to a smooth function we will need some estimates on these functions and their Laplacian powers. What we know so far is that they are finite linear combinations of the localized eigenfunctions $u_{i,m}$ with $0 \le m \le j$ and $i = 1, 2$, and the generalized eigenfunctions E_λ^k for a fixed λ and $0 \le k \le j$. All of these functions and their Laplacian powers of order at most j are bounded: for the localized eigenfunctions this is obvious, while for the generalized eigenfunctions it follows from the bound in Theorem 1.1 and the recursion (4.3). We conclude that for each j there is a constant $C(j)$ such that for all $0 \le k \le j$

$$|\Delta^k F_j| \le C(j) \qquad\qquad |\Delta^k G_j| \le C(j) \qquad\qquad |\Delta^k H_j| \le C(j)$$

and turn now to a scaling argument that allows us to make these as small as desired.

Lemma 3.4. *If $m \in \mathbb{N}$ then the functions*

$$F_{j,m} = 5^{-jm} F_j \circ f_0^{-m} \qquad G_{j,m} = 5^{-jm} G_j \circ f_0^{-m} \qquad H_{j,m} = 5^{-jm} H_j \circ f_0^{-m}$$

have the same j-jets at q_0 as F_j, G_j and H_j respectively, but for $0 \leq k \leq j$ they satisfy the following estimates on SG_∞

$$|\Delta^k F_{j,m}| \leq C(j) 5^{(k-j)m} \qquad |\Delta^k G_{j,m}| \leq C(j) 5^{(k-j)m} \qquad |\Delta^k H_{j,m}| \leq C(j) 5^{(k-j)m}.$$

Proof. The result is an elementary consequence of the scaling property of the Laplacian. By induction from (2.4) we see that $\Delta^k(u \circ f_0^{-m}) = 5^{km}(\Delta^k u) \circ f_0^{-m}$. Both statements of the lemma are immediate consequences of this and the definitions (2.5) and (2.6) of the normal and tangential derivatives. □

Proof of Theorem 1.2. We are supplied with values $(\zeta_k, \eta_k, \theta_k)$ for $k \in \mathbb{N}$ and seek a smooth function u such that $\Delta^k u(q_0) = \zeta_k$, $\partial_n \Delta^k u(q_0) = \eta_k$ and $\partial_T \Delta^k u(q_0) = \theta_k$ for all k. This will be certainly be the case for the function

$$u = \sum_{j=0}^\infty \left(\zeta_j F_{j,m_j} + \eta_j G_{j,m_j} + \theta_j H_{j,m_j} \right) \tag{3.1}$$

provided only that applying any power of the Laplacian yields a uniformly convergent series. However by Lemma 3.4 we may choose the sequence m_j such that terms after the j-th have only a small effect on the Laplacian powers of order at most j. Specifically, given any $\epsilon > 0$ we may make m_j so large that for $0 \leq k \leq j-1$

$$\left| \Delta^k \left(\zeta_j 5^{-jm_j} F_{j,m_j} + \eta_j 5^{-jm_j} G_{j,m_j} + \theta_j 5^{-jm_j} H_{j,m_j} \right) \right|$$
$$\leq C(j) 5^{(k-j)m_j} \max\{|\zeta_k|, |\eta_k|, |\theta_k| : 0 \leq k \leq j-1\}$$
$$\leq \epsilon 2^{k-j}$$

providing a bound on the tail of the series obtained by applying Δ^k to (3.1). We conclude that u is smooth, that it has the desired jet at q_0, and moreover that most of the contribution to the k-th jet is from the first k terms in the sum:

$$\left| \Delta^k u \right| \leq \epsilon + C(k) \sum_{j=0}^k (|\zeta_j| + |\eta_j| + |\theta_j|).$$

□

4 Generalized eigenfunctions with decay

In this section we prove Theorem 1.1, showing that there are exponentially decaying generalized eigenfunctions E_λ^j of the Laplacian on SG_∞. Our results depend on work in [7], where the negative-eigenvalue eigenfunctions of $-\Delta$ were studied using spectral decimation. Using notation from (2.7), the results we need may be summarized as:

Proposition 4.1 ([7], Section 6). *For each $\lambda < 0$ there is an eigenfunction E_λ on SG_∞ which is symmetrical under the reflection that fixes q_0 and exchanges q_1 with q_2, and which satisfies $(\Delta + \lambda)E_\lambda = 0$. There is an explicit formula for E_λ at the points $z_m = f_0^{-m}(q_1)$*

$$E_\lambda(z_m) = 1 - \frac{\lambda_m}{4} + \frac{\lambda_m}{4} \prod_{j=0}^{\infty}\left(1 + \frac{4}{2 - \lambda_{m-j}}\right) \tag{4.1}$$

which is uniformly continuous on compacta in V_m, and E_λ is the limit of this on SG_∞. These functions have exponential decay $|E_\lambda(z_m)| = O(|\lambda_m|^{-1}) = O\left(2^{-2^{-m}}\right)$ as $m \to -\infty$ and the normal derivative at q_0 is given by

$$\partial_n E_\lambda(q_0) = \lambda \prod_{m=0}^{\infty}\left(1 + \frac{4}{2 - \lambda_{-m}}\right) \prod_{n=1}^{\infty}\left(\frac{6 - \lambda_n}{6 - 3\lambda_n}\right) > 0. \tag{4.2}$$

Our construction is motivated as follows. Formally setting $E_\lambda^j = (\frac{d}{d\lambda})^j E_\lambda$ we find that the E_λ^j satisfy the generalized eigenfunction equation

$$(\Delta + \lambda)E_\lambda^j = -jE_\lambda^{j-1} \tag{4.3}$$

and we hope that the decay of E_λ will ensure exponential decay for E_λ^j. This argument is made rigorous by Lemma 4.2, but it will initially be simpler to construct the E_λ^j from (4.3) than by proving E_λ can be differentiated with respect to λ.

Observe that on SG we can inductively obtain solutions E_λ^j of (4.3) for $j \in \mathbb{N}$ starting with $E_\lambda^0 = E_\lambda$, merely because $\lambda < 0$ and the spectrum of Δ is positive. The resulting functions are clearly in $\mathrm{dom}(\Delta^\infty)$, and depend on the boundary data we assign. Guided by the formal idea that E_λ^j should be $(\frac{d}{d\lambda})^j E_\lambda$ we set

$$E_\lambda^j(z) = \left(\frac{d}{d\lambda}\right)^j E_\lambda(z) \tag{4.4}$$

at each of the three boundary points $z = q_0, q_1, q_2$. The definition is legitimate because $\lambda_m = \Psi(5^{-m}\lambda)$ and Ψ is entire (Lemma 2.1), so the rapid growth of λ_m ensures the expression (4.1) is analytic with respect to λ.

Lemma 4.2. *Using the supremum norm, the functions E_λ^j are differentiable with respect to λ and $\frac{d}{d\lambda}E_\lambda^{j-1} = E_\lambda^j$.*

Proof. We require a standard estimate (like Lemma 5.2.8 of [5]). Let u be in $\mathrm{dom}(\Delta)$ and subtract the harmonic function u_0 with the same boundary values. It is well known that then $\|u - u_0\|_\infty \leq c\|\Delta u\|_2$, and by the maximum principle we conclude $\|u\|_\infty \leq c\|\Delta u\|_2 + \max_{V_0}|u|$. Let $\lambda < 0$, and let $\kappa_1 > 0$ be the first Dirichlet eigenvalue of $-\Delta$. The spectral representation of Δ immediately shows

$$\|u\|_\infty \leq c\left(1 + \frac{|\lambda|}{\kappa_1}\right)\left\|(\Delta + \lambda)u\right\|_2 + \max_{V_0}|u|. \tag{4.5}$$

9

Now suppose inductively that the lemma is true up to $j - 1$. For the difference between E_λ^j and the Newton quotient for the derivative of E_λ^{j-1} we have

$$(\Delta + \lambda)\Big(E_\lambda^j - \frac{1}{t}(E_{\lambda+t}^{j-1} - E_\lambda^{j-1})\Big)$$

$$= -jE_\lambda^{j-1} - \frac{1}{t}\Big(-(j-1)E_{\lambda+t}^{j-2} - tE_{\lambda+t}^{j-1} + (j-1)E_\lambda^{j-2}\Big)$$

$$= (E_{\lambda+t}^{j-1} - E_\lambda^{j-1}) + (j-1)\frac{1}{t}\Big(E_{\lambda+t}^{j-2} - E_\lambda^{j-2}\Big) - (j-1)E_\lambda^{j-1}$$

$$\to 0 \quad \text{in } L^2(SG)$$

by induction. From (4.5) and the fact that the boundary data varies analytically with λ we conclude

$$\Big\|E_\lambda^j - \frac{1}{t}(E_{\lambda+t}^{j-1} - E_\lambda^{j-1})\Big\|_\infty \to 0.$$

The same reasoning reduces the base case of the induction to showing $\|E_{\lambda+t} - E_\lambda\|_2 \to 0$, which is a consequence of the fact (from Proposition 4.1) that E_λ is uniformly approximated by the analytic function of λ in (4.1). $\qquad\square$

With this in hand we can describe the natural scaling behavior of the E_λ^j. From (2.4) we know that $E_{5\lambda} = E_\lambda \circ f_0^{-1}$, whence on $f_0(SG)$

$$E_\lambda^j \circ f_0^{-1} = \Big(\frac{d}{d\lambda}\Big)^j E_\lambda \circ f_0^{-1} = \Big(\frac{d}{d\lambda}\Big)^j E_{5\lambda} = 5^j E_{5\lambda}^j$$

and therefore the natural definition of E_λ^j on $f_0^{-1}(SG)$ is to set $E_\lambda^j = 5^j E_{5\lambda}^j \circ f_0$. Inductively we let

$$E_\lambda^j = 5^{jn} E_{5^n\lambda} \circ f_0^n \quad \text{on } f_0^{-n}(SG) \tag{4.6}$$

for each $n \in \mathbb{N}$ to extend E_λ^j to all of SG_∞. We remark that this gives the same result as solving (4.3) on $f_0^{-n}(SG)$ with boundary data (4.4) at the points $z = q_0, f_0^{-n}(q_1), f_0^{-n}(z_2)$.

Combining the above results we have proven the existence statements of Theorem 1.1. What remains to be proven is the content of the following lemma, which is regrettably but perhaps unavoidably technical.

Lemma 4.3. *The generalized eigenfunctions satisfy $|E_\lambda^j| \le j!|\lambda|^{-j}$ and have decay*

$$\Big|E_\lambda^j(z_m)\Big| = O(2^{-2^{-m}}) \quad \text{as } m \to -\infty.$$

Proof. We first prove the decay. Recall that the recursion (2.7) guarantees $|\lambda_m| \ge C2^{2^{-m}}$ as $m \to -\infty$. Set $\beta_m^j = (d/d\lambda)^j \lambda_m$. It is elementary to verify that $|\beta_m^j| = O(|\lambda_m|)$ as $m \to -\infty$ from the definition of Ψ and the recursion (2.10). Using the explicit formula (4.1) we write $E_\lambda(z_m) = 1 + (P_m - \lambda_m)/4$, where

$$P_m = \lambda_m \prod_{n=0}^{\infty} \Big(1 + \frac{4}{2 - \lambda_{m-n}}\Big)$$

10

and define S_m^j by $(d/d\lambda)^j P_m = P_m S_m^j$. Examining S_m^1 we have

$$
\begin{aligned}
S_m^1 &= \frac{\beta_m^1}{\lambda_m} + \sum_{n=0}^{\infty} \frac{4\beta_{m-n}^1}{(2 - \lambda_{m-n})(6 - \lambda_{m-n})} \\
&= \frac{\beta_m^1}{\lambda_m} + \frac{4\beta_m^1}{\lambda_m^2} + O(\lambda_m^{-2}) + \sum_{n=1}^{\infty} \frac{4\beta_{m-n}^1}{(2 - \lambda_{m-n})(6 - \lambda_{m-n})} \\
&= \frac{\beta_m^1}{\lambda_m} + \frac{4\beta_m^1}{\lambda_m^2} + O(\lambda_m^{-2})
\end{aligned}
$$

where the penultimate estimate uses the series expansion for $1/(2 - \lambda_m)(6 - \lambda_m)$, and the final one uses the structure of the series. This series consists of terms which are rational functions of λ_{m-n} and β_{m-n}^1, and in which the degree of the denominator strictly exceeds that of the numerator. The rapid growth of $|\lambda_m|$ ensures this is bounded by a multiple of the first term, which is $O(|\lambda_{m-1}|^{-1}) = O(|\lambda_m|^{-2})$.

We will call a series in which the terms are rational functions that depend on $\lambda_{m-n}, \beta_{m-n}^1, \ldots, \beta_m^j$ for $n \geq 1$, but must always have the degree of the denominator to be strictly greater than the numerator, a good rational series or GRS. Notice that the derivative of a GRS is a GRS, and that the product of a GRS with a GRS or a rational function in which the numerator has the same or lesser degree than the denominator is also a GRS. Our estimates on λ_{m-n} and β_{m-n}^1 guarantee that any GRS sums to a value which is $O(|\lambda_m|^{-2})$.

Using induction over j we see that the function $S_m^j - \beta_m^j/\lambda_m - 4\beta_m^j/\lambda_m^2$ is always a GRS, so is $O(|\lambda_m|^{-2})$. Indeed, this is true for $j = 1$, and if we assume it to be true for $j - 1$ and apply the recursion $S_m^j = S_m^1 S_m^{j-1} + (dS_m^{j-1}/d\lambda)$, then both

$$
\begin{aligned}
S_m^1 S^{j-1,m} &= \left(\frac{\beta_m^1}{\lambda_m} + \frac{4\beta_m^1}{\lambda_m^2} + \text{GRS} \right)\left(\frac{\beta_m^{j-1}}{\lambda_m} + \frac{4\beta_m^{j-1}}{\lambda_m^2} + \text{GRS} \right) \\
&= \frac{\beta_m^1 \beta_m^{j-1}}{\lambda_m^2} + \frac{8\beta_m^1 \beta_m^{j-1}}{\lambda_m^3} + GRS
\end{aligned}
$$

and

$$
\frac{dS_{j-1,m}}{d\lambda} = \frac{\beta_m^j}{\lambda_m} - \frac{\beta_m^{j-1}\beta_m^1}{\lambda_m^2} + \frac{4\beta_m^j}{\lambda_m^2} - \frac{8\beta_m^{j-1}\beta_m^1}{\lambda_m^3} + GRS
$$

so we may sum them to complete the induction. In particular we conclude that

$$
\begin{aligned}
4E_\lambda^j(z_m) &= \frac{d^j P_m}{d\lambda^j} - \beta_m^j \\
&= P_m S_m^j - \beta_m^j \\
&= P_m\left(S_m^j - \frac{\beta_m^j}{\lambda_m} - \frac{4\beta_m^j}{\lambda_m^2} \right) + \frac{\beta_m^j}{\lambda_m}(P_m - \lambda_m + 4) + \frac{4\beta_m^j}{\lambda_m^2}(P_m - \lambda_m) \\
&= O(|\lambda_m|^{-1}) = O(2^{-2^{-m}})
\end{aligned}
$$

11

where in the last step we used that the first bracketed term is $O(|\lambda_m|^{-2})$, and that $P_m = O(|\lambda_m|)$, $(P_m - \lambda_m + 4) = O(|\lambda_m|^{-1})$ and $(P_m - \lambda_m) = O(1)$, all of which are from the fact that $|E_\lambda(z_m)| = O(|\lambda_m|^{-1})$ (see Proposition 4.1) and the definition of P_m.

Now that we know $|E_\lambda^j|$ has exponential decay it must be the case that its maximum occurs at an interior point of some $f_0^{-m}(SG)$. It is well known (see [7] Proposition 2.11) that E_λ^j and ΔE_λ^j must have opposite signs at any local extreme point of $|E_\lambda^j|$. Since $\Delta E_\lambda^j = -\lambda E_\lambda^j - j E_\lambda^{j-1}$ and $\lambda < 0$ we find that $\operatorname{sgn}(\Delta E_\lambda^j) = -\operatorname{sgn}(E_\lambda^j)$ implies $|\lambda E_\lambda^j| \leq |j E_\lambda^{j-1}|$. The bound on E_λ^j follows by induction and the fact that $|E_\lambda| \leq |E_\lambda(q_0)| = 1$. □

References

[1] Martin T. Barlow and Jun Kigami, *Localized eigenfunctions of the Laplacian on p.c.f. self-similar sets*, J. London Math. Soc. (2) **56** (1997), no. 2, 320–332. MR MR1489140 (99b:35162)

[2] Nitsan Ben-Gal, Abby Shaw-Krauss, Robert S. Strichartz, and Clint Young, *Calculus on the sierpinski gasket II: Point singularities, eigenfunctions, and normal derivatives of the heat kernel*, Trans. Amer. Math. Soc., posted on April 17, 2006, PII: S 0002-9947(06)04056-6 (to appear in print).

[3] Kyallee Dalrymple, Robert S. Strichartz, and Jade P. Vinson, *Fractal differential equations on the Sierpinski gasket*, J. Fourier Anal. Appl. **5** (1999), no. 2-3, 203–284. MR MR1683211 (2000k:31016)

[4] M. Fukushima and T. Shima, *On a spectral analysis for the Sierpiński gasket*, Potential Anal. **1** (1992), no. 1, 1–35. MR MR1245223 (95b:31009)

[5] Jun Kigami, *Analysis on fractals*, Cambridge Tracts in Mathematics, vol. 143, Cambridge University Press, Cambridge, 2001. MR MR1840042 (2002c:28015)

[6] Leonid Malozemov and Alexander Teplyaev, *Self-similarity, operators and dynamics*, Math. Phys. Anal. Geom. **6** (2003), no. 3, 201–218. MR MR1997913 (2004d:47012)

[7] Jonathan Needleman, Robert S. Strichartz, Alexander Teplyaev, and Po-Lam Yung, *Calculus on the Sierpinski gasket. I. Polynomials, exponentials and power series*, J. Funct. Anal. **215** (2004), no. 2, 290–340. MR MR2150975

[8] Luke G. Rogers, Robert S. Strichartz, and Alexander Teplyaev, *Smooth bumps, a Borel theorem and partitions of unity on p.c.f. fractals*, In preparation.

[9] Robert S. Strichartz, *Differential equations on fractals: A tutorial*, Princeton University Press, to appear 2006.

[10] _____, *Fractals in the large*, Canad. J. Math. **50** (1998), no. 3, 638–657. MR MR1629847 (99f:28015)

[11] Alexander Teplyaev, *Spectral analysis on infinite Sierpiński gaskets*, J. Funct. Anal. **159** (1998), no. 2, 537–567. MR MR1658094 (99j:35153)