Abstract

A Degree-Independent Sobolev Extension Operator

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We consider a domain $\Omega \subset \mathbb{R}^n$ and the Sobolev spaces $W^{k,p}$ of functions with k derivatives in L^p. It is well known that extension operators from $W^{k,p}(\Omega)$ to $W^{k,p}(\mathbb{R}^n)$ exist only under some assumptions on the geometry of Ω. In the case that Ω has Lipschitz boundary, Calderón showed that for each integer k there is an extension operator valid on $W^{k,p}(\Omega)$ for $1 < p < \infty$. Later work of Stein introduced a degree-independent operator for a Lipschitz domain, so that a single operator could be used on $W^{k,p}(\Omega)$ for all integer k and all $1 \leq p \leq \infty$. Subsequently Jones introduced an extension operator on locally uniform domains. This is a much larger class of domains that includes examples with highly non-rectifiable boundaries. Jones also proved that these are the sharp class of domains for extension of Sobolev spaces in $\mathbb{R}^2$. The operators constructed by Jones are degree-dependent: the extension operator for $W^{k,p}(\Omega)$ is not defined on spaces with lower degrees of smoothness. In the present work we extend the methods used by Stein and Jones and thereby produce a degree-independent operator that may be used on all spaces $W^{k,p}(\Omega)$ on a locally uniform domain Ω.

A Degree-Independent Sobolev Extension Operator

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Chapter 1

Preliminaries

1.1 Definitions and Notation

Balls, Cubes and the Dyadic Grid

We work on the n-dimensional Euclidean space $\mathbb{R}^n$ and on an open connected domain $\Omega \subset \mathbb{R}^n$. Points are denoted x or $(x_1, x_2, \ldots, x_n)$. The Euclidean distance between two points is $|x-y|$, the distance from x to a set A is $\operatorname{dist}(x, A) = \inf_{y\in A} |x-y|$, and the distance between two sets is $\operatorname{dist}(A, B) = \inf |x-y| : x \in A, y \in B$. Balls are written $B(x, r) = \{y : |x-y| \leq r\}$. At times it will be convenient to write λB for the ball concentric with B but having λ times its radius.

A set of the form $Q_l(x) = \{y : |y_j - x_j| \leq l/2\}$ is called a cube of center x and size or sidelength l. Usually the center of the cube Q is denoted x_Q and its size is $l(Q)$. As with balls, λQ is the cube with the same center as Q but size λ times as large. A dyadic cube of scale $2^j, j \in \mathbb{Z}$, is a cube having size 2^j and all of whose vertices lie on the lattice $(2^j\mathbb{Z})^n$. Clearly each dyadic cube of scale 2^j can be divided into 2^n dyadic cubes of scale 2^{j-1} (called its dyadic children) and is itself contained in a unique cube of scale 2^{j+1} (its dyadic parent). The useful covering properties of the dyadic grid of cubes arise as a result of the following observation: if $Q_j \neq Q_k$ are dyadic cubes of any scale then either they have disjoint interiors or the smaller is contained in the larger. Given a collection of dyadic cubes we can then obtain a cover of their union in which all boxes have pairwise disjoint interiors by merely removing from the collection any box which is contained in some larger box. The remaining boxes are those which were maximal under inclusion, and by the above observation they have disjoint interiors.

The Whitney Decomposition

It is a result of Whitney that any open set $\Omega \subset \mathbb{R}^n$ may be decomposed into a collection of dyadic cubes Q_j such that $l(Q_j)$ is comparable to the distance of Q_j from $\partial\Omega$. The proof we use is from Stein [Ste70] Chapter VI, Section 1.

Lemma 1.1.1. *If $\Omega \subset \mathbb{R}^n$ is open then there is a countable collection $\{Q_j\}$ of dyadic cubes with disjoint interiors such that*

$$1 \leq \frac{\operatorname{dist}(Q_j, \partial\Omega)}{\sqrt{n}l(Q_j)} \leq 4 \tag{1.1}$$

and if $Q_j \bigcap Q_k \neq \emptyset$

$$\frac{1}{4} \leq \frac{l(Q_j)}{l(Q_k)} \leq 4. \tag{1.2}$$

The collection $\mathcal{W} = \{Q_j\}$ is called the Whitney decomposition of Ω.

Proof. Consider for each $j \in \mathbb{Z}$ the collection $\mathcal{V}$ of dyadic cubes of length 2^j that have non-empty intersection with the set $\Omega_j = \{x : 2^{j+1}\sqrt{n} < \operatorname{dist}(x, \partial\Omega) \leq 2^{j+2}\sqrt{n}\}$. It is clear that $\Omega = \cup\Omega_j$ and that every $x \in \Omega_j$ is contained in a dyadic box of length 2^j, whereupon the cubes in $\mathcal{V}$ cover Ω. Moreover we have

$$\operatorname{dist}(Q, \partial\Omega) \leq \operatorname{dist}(x, \partial\Omega) \leq 2^{j+2}\sqrt{n} = 4l(Q)$$

$$\operatorname{dist}(Q, \partial\Omega) \geq 2^{j+1}\sqrt{n} - \operatorname{diam}(Q) = 2^{j+1}\sqrt{n} - 2^{j}\sqrt{n} = 2^{j}\sqrt{n} = l(Q)\sqrt{n}$$

so that we have verified condition (1.1). It also follows that these cubes do not intersect Ω^c. In order to obtain cubes with disjoint interiors and condition (1.2) we take $\mathcal{W}$ to be the subcollection of cubes of $\mathcal{V}$ which are maximal under inclusion. These cubes cover Ω and have disjoint interiors. We now know that $Q_1, Q_2 \in \mathcal{W}$ can only intersect if they have a common boundary point, in which case we can apply (1.1) to obtain

$$\operatorname{dist}(Q_2, \partial\Omega) \leq \operatorname{dist}(Q_1, \partial\Omega) + \sqrt{n}l(Q_1) \leq 5l(Q_1)$$

but since $l(Q_2) = 2^{j}l(Q_1)$ for some j we deduce $j = 2$ and thereby establish (1.2). □

Observe also that a Whitney cube containing a point of known distance to $\partial\Omega$ cannot be too small. In particular if $x \in Q$ and Q is a Whitney cube then by (1.1)

$$4\sqrt{n}l(Q) \geq \operatorname{dist}(Q, \partial\Omega) \geq \operatorname{dist}(x, \partial\Omega) - \sqrt{n}l(Q)$$

and therefore we have

Lemma 1.1.2. *If Q is the Whitney cube of Ω containing x then*

$$l(Q) \geq \frac{\operatorname{dist}(x, \partial\Omega)}{5\sqrt{n}}$$

Lebesgue and Sobolev Spaces

We use $L^p(\Omega, dx)$ to denote the Lebesgue spaces of (equivalence classes of) functions on Ω with

$$||f||_{L^p(\Omega,dx)} = \left(\int_\Omega |f|^p\, dx\right)^{1/p} < \infty, \quad \text{if } 0 < p < \infty$$
$$||f||_{L^\infty(\Omega,\, dx)} = \operatorname{esssup}_\Omega |f| < \infty, \quad \text{if } p = \infty$$

where dx is Lebesgue measure. If no domain is mentioned it is assumed to be all of $\mathbb{R}^n$.

Given a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ of length $|\alpha| = \sum_j \alpha_j$ we write D^α for the derivative $(\partial/\partial x_1)^{\alpha_1} \ldots (\partial/\partial x_n)^{\alpha_n}$. A locally integrable function f on Ω is said to have a weak α-derivative if there is another locally integrable function which we denote $D^\alpha f$ and which satisfies the identity

$$\int_\Omega (D^\alpha f)\phi = (-1)^{|\alpha|} \int_\Omega f(D^\alpha \phi)$$

for all $C^\infty(\Omega)$ functions ϕ which have compact support in Ω. The function f is k times weakly differentiable (for $k \in \mathbb{N}$) if it has weak derivatives $D^\alpha f$ for all $|\alpha| \le k$. The weak gradient is the vector $\nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$ and ∇^k is the vector of all weak partial derivatives D^α of order $|\alpha| = k$.

A function f which is k times weakly differentiable on Ω is an element of the Sobolev space $W^{k,p}(\Omega)$ if it has finite Sobolev norm:

$$||f||_{W^{k,p}(\Omega)} = \sum_{|\alpha| \le k} ||D^\alpha f||_{L^p(\Omega)} < \infty.$$

For ease of notation we may at times wish to refer to the value $f(x)$ of some $f \in W^{k,p}(\Omega)$ at a point $x \in \Omega$. This is not an a-priori well defined quantity, so we make the usual convention that at the Lebesgue points of f

$$f(x) = \lim_{r \to 0} \fint_{B(x,r)} f$$

where $\fint f$ denotes the average of f. At the points which are not Lebesgue we set $f(x) = 0$.

Lipschitz Domains

A Special Lipschitz Domain ([Ste70] Chapter VI, Section 3.2) is a set of points lying above a Lipschitz graph. More precisely, let $A(x)$ be a real-valued function on $\mathbb{R}^{n-1}$ which is Lipschitz, i.e.

$$||A||_{\text{Lip}} = \sup \frac{|A(x) - A(y)|}{|x - y|} < \infty$$

and define a domain $\Omega_A = \{(x, x_n) \in \mathbb{R}^n : x_n > A(x),\ x \in \mathbb{R}^{n-1}\}$. Any set which may be rotated to coincide with a domain of the form Ω_A is called a Special Lipschitz Domain.

A Lipschitz domain is a domain whose boundary consists of a union of Lipschitz pieces, no one of which has too small a diameter. One way to define such a domain Ω (depending on parameters $\delta > 0$, $J \in \mathbb{Z}$ and a Lipschitz bound $M > 0$) is to require that there be a countable collection of balls $\{B(x_j, \delta)\}$ which have the following properties:

1. No point of $\mathbb{R}^n$ is contained in more than J distinct balls from the collection.

2. The balls cover a $(\delta/2)$-neighborhood of $\partial\Omega$, so that any x with $\text{dist}(x, \partial\Omega) < \delta/2$ is in $\cup_j B_j$.

3. For each j there is a Lipschitz map A_j such that the set $\Omega \cap B_j$ may be translated and rotated to coincide with $\Omega_{A_j} \cap B(0, \delta)$.

This definition is from [Ste70] Chapter VI, Section 3.3, where these are called *minimally smooth* domains. A large number of different domains are considered in the literature under various different names. For example, Adams calls the above condition the *Strong Local Lipschitz Condition* (see [AF03], page 83). Both Adams ([AF03]) and Maz′ya ([MP97]) extensively discuss conditions of this type, as well as conditions involving cones (described below) at boundary points. In the interests of brevity we will ignore all of these distinctions, even when doing so slightly weakens the statements of known theorems. We do mention that many results which are true of Sobolev spaces on special Lipschitz domains may be transferred to Lipschitz domains via an appropriate smooth partition of unity. A proof that this is the case for the types of problems considered in this thesis may be found in [Ste70] Chapter VI Section 3.3, but we shall not repeat it here.

From our perspective, one of the most useful features of Lipschitz domains is the existence of cones at boundary points. We call any set which may be rotated to coincide with

$$\Gamma(\alpha,\delta) = \{(x, x_n) : |x| \leq |x_n| \tan\alpha,\ 0 \leq x_n \leq \delta,\ x \in \mathbb{R}^{n-1}\} \tag{1.3}$$

a cone of length δ, angle α and vertex at the origin. We let $\Gamma^- = \{(x, -x_n) : (x, x_n) \in \Gamma\}$ and define a double cone to be any set obtained by rotation of $\tilde{\Gamma} = \Gamma \cup \Gamma^-$. Given a Lipschitz domain it is not difficult to define a collection $\{\tilde{\Gamma}_j\}$ of rotations of a fixed double cone with vertex at the origin, length and angle depending on δ and M, and such that at every point $x \in \partial\Omega$ we have a double cone $\tilde{\Gamma}_j = \Gamma \cup \Gamma^-$ with

$$\{x + y : y \in \Gamma_j \setminus 0\} \subset \Omega$$

$$\{x + y : y \in \Gamma_j^- \setminus 0\} \subset \Omega^c.$$

Locally Uniform Domains

Locally uniform domains were introduced by Martio and Sarvas [MS79] and have been extensively studied. In [Jon80] Jones identified these domains as the extension domains

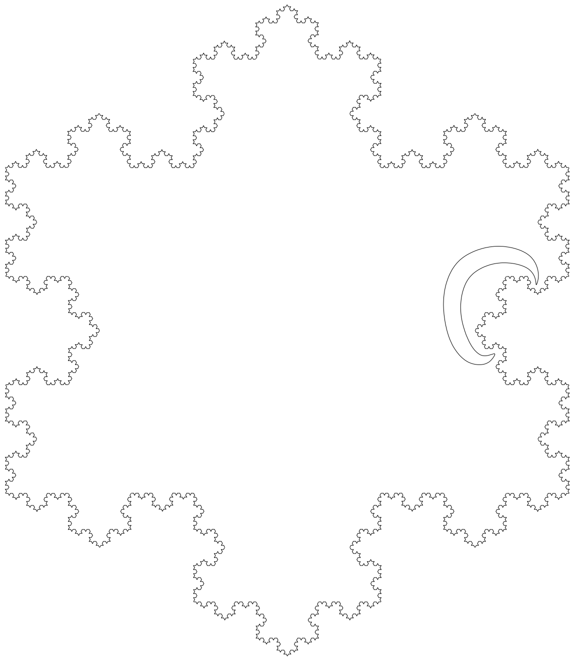

Figure 1.1: The Koch snowflake is locally uniform

for BMO functions, and in [Jon81] he addressed the question of extension of Sobolev functions on these domains (see also Theorem 1.3.3). Jerison and Kenig [JK82] studied potential theory on locally uniform domains; their work and that of later authors showed that these are essentially the largest class of domains on which there is a theory of potentials analogous to that for the upper half space.

Several different definitions of locally uniform domains occur in the literature. We use the one found in [Jon81].

Definition 1.1.3. *A domain is* (ϵ, δ)*-locally uniform if between any pair of points x,y such that* $|x - y| < \delta$ *there is a rectifiable arc* $\gamma \subset \Omega$ *of length at most* $|x - y|/\epsilon$ *and having the property that for all* $z \in \gamma$

$$\mathrm{dist}(z, \partial\Omega) \geq \frac{\epsilon|z - x||z - y|}{|x - y|}. \tag{1.4}$$

It is easy to see that a Lipschitz domain is locally uniform for some values of ϵ and δ.

An example of a locally uniform domain which is not Lipschitz is the interior of the Koch snowflake (Figure 1.1). The boundary of this set is not only non-rectifiable but indeed not of Hausdorff dimension 1. In general it is possible to define for any $\lambda \in [n-1, n)$ a locally uniform domain in $\mathbb{R}^n$ with boundary dimension λ, however it is not possible that the boundary have positive measure in $\mathbb{R}^n$.

Lemma 1.1.4. *If $\Omega \subset \mathbb{R}^n$ is (ϵ, δ) locally uniform then the n-dimensional Lebesgue measure of the boundary is $|\partial\Omega| = 0$.*

Proof. Fix $y \in \partial\Omega$. We show it cannot be a Lebesgue density point of $\partial\Omega$. For $r > 0$ consider $B(y, r)$. If $r < \delta$ is sufficiently small then there is $x_1 \in \Omega \cap \partial B(x, r)$ and $x_2 \in \Omega \cap B(x, r/4)$. By the definition of local uniformity there is an arc joining them and therefore a point $z \in \Omega \cap \partial B(x, r/2)$. We have

$$\frac{|x_1 - z||x_2 - z|}{|x_1 - x_2|} \geq \frac{r}{8}$$

and so $\operatorname{dist}(z, \partial\Omega) \geq \epsilon r/8$ by (1.4). Applying Lemma 1.1.2 implies the Whitney cube $Q \ni z$ has $l(Q) \geq \dfrac{\epsilon r}{40\sqrt{n}}$, and hence that the proportion of $B(y, r)$ that is contained in Ω is bounded below. It follows that y cannot be a density point of $\partial\Omega$ and therefore $|\partial\Omega| = 0$. □

1.2 An Extension Problem

If $\Omega \subset \mathbb{R}^n$ is a domain and $f \in W^{k,p}(\mathbb{R}^n)$ then it is obvious that the restriction of f to Ω is in $W^{k,p}(\Omega)$. A natural question to ask is when the converse is the case. The following example shows that this may depend on the geometry of $\partial\Omega$, and in particular that an outward cusp may restrict the spaces that can be extended.

Example 1.2.1. *Not all functions from $W^{1,2+\epsilon}(\Omega)$ may be extended to $W^{1,2+\epsilon}(\mathbb{R}^2)$ if Ω is the* *set*

$$\Omega = \left\{(x, y) \in \mathbb{R}^2 : 0 \le y \le x^{1+3\epsilon},\ 0 \le x \le 1\right\}$$

This domain is illustrated in Figure 3.4.2.

To see that this is the case first notice that any function from $W^{1,2+\epsilon}(\mathbb{R}^2)$ is continuous by the Sobolev embedding theorem. However the function $f(x, y) = x^{-\epsilon/(2+\epsilon)}$ has

$$\int_\Omega |f(z)|^{2+\epsilon} = \int_0^1 x^{-\epsilon} x^{1+3\epsilon}\, dx = \frac{1}{2+2\epsilon}$$
$$\int_\Omega |\nabla f(z)|^{2+\epsilon} = \left(\frac{\epsilon}{2+\epsilon}\right)^{2+\epsilon} \int_0^1 x^{-2-2\epsilon} x^{1+3\epsilon}\, dx = \left(\frac{\epsilon}{2+\epsilon}\right)^{2+\epsilon} \frac{1}{\epsilon}$$

and it follows that f is in $W^{1,2+\epsilon}(\Omega)$. Clearly f has no continuous extension to $\mathbb{R}^2$ and therefore no extension in $W^{1,2+\epsilon}(\mathbb{R}^2)$.

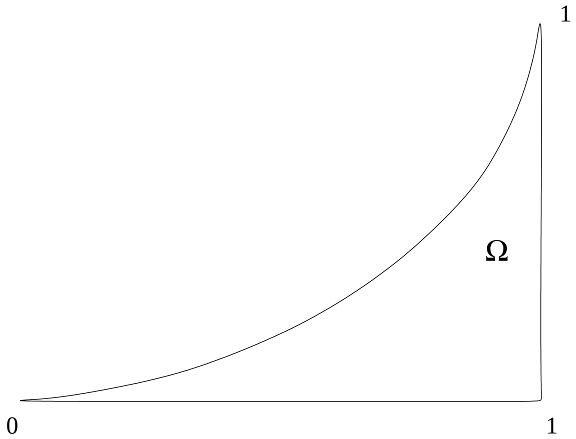

Figure 1.2: A domain for which extension is not possible

For finitely connected $\Omega \subset \mathbb{R}^2$ it is shown in [Jon81] that the presence of cusps on the boundary of Ω is exactly what obstructs extension of Sobolev functions. The precise result is given below as part of Theorem 1.3.4, while the nature of the obstruction caused by a cusp is further explored at the beginning of Chapter 3.

For the remainder of this thesis we shall be interested in conditions on the domain Ω that guarantee all (or most) of the spaces $W^{k,p}(\Omega)$ arise precisely as restrictions of $W^{k,p}(\mathbb{R}^n)$, and on the construction of operators of the form

$$\mathcal{E} : W^{k,p}(\Omega) \longrightarrow W^{k,p}(\mathbb{R}^n)$$

with estimates

$$\|\mathcal{E}f\|_{W^{k,p}(\mathbb{R}^n)} \leq C\|f\|_{W^{k,p}(\Omega)}$$

1.3 Historical Remarks

Early Results

The problem of how to extend Sobolev functions was recognized early in the development of the theory, but it is fair to say that the particular variant we are interested in was not the focus of attention. Instead many people were interested in determining the trace space of the Sobolev space to the boundary of a domain and the circumstances under which functions from the trace space might be extended. In this direction we mention in particular the works of Sobolev ([Sob50, Sob63]) and of Deny and Lions ([DL54]), which addressed the important special case of the trace of $W^{1,2}$ functions. The first result for more general Sobolev functions is in a paper of Gagliardo ([Gag57]), who identified the trace of the spaces $W^{1,p}$ for all $1 \leq p \leq \infty$. All of these results were for domains with Lipschitz boundary. We also mention the works of Slobodeckiĭ ([Slo58]), Aronszajn and Smith ([AS61]), Lizorkin ([Liz62]), and Stein ([Ste62]), all of which appeared more or less contemporaneously with the results of Calderón discussed below.

Calderón, Stein, and Jones

The first extension theorem that considered all spaces $W^{k,p}$ is due to Calderón [Cal61] and was an outgrowth of his work on Bessel potentials. He considered a class of domains that is slightly more general than the Lipschitz domains defined earlier, and used a defintion written in terms of cones rather than Lipschitz graphs. We will not give a precise definition of these domains, but state the following theorem as a consequence of his result.

Theorem 1.3.1 (Calderón,). *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain. For each fixed $k \in \mathbb{N}$ there is a bounded linear extension operator such that for all $1 < p < \infty$*

$$\mathcal{E}_C^k : W^{k,p}(\Omega) \longrightarrow W^{k,p}(\mathbb{R}^n) \tag{1.5}$$

with bound depending on n, p, k, and the constants of the Lipschitz domain. This extension has the further property that it extends any $f \in W_0^{k,p}(\Omega)$ to be zero outside Ω.

The operator $\mathcal{E}_C^k$ is given by an explicit formula involving integration of f against a singular kernel supported on a cone (as defined in (1.3)). The constraint $1 < p < \infty$ is due to the use of the Calderón-Zygmund theory of singular integrals in the proof. It is worth remarking that since the operator extends functions from $W_0^{k,p}(\Omega)$ to be zero outside Ω we might expect that it may be interpreted in terms of an extension from a function space defined on $\partial\Omega$. This is indeed the case and is a particular strength of this theorem as it actually identifies the trace of $W^{k,p}(\Omega)$ to $\partial\Omega$.

Observe that Theorem 1.3.1 really proves the existence of an infinite collection of operators $\mathcal{E}_C^k$, one for each $k \in \mathbb{N}$. In [Ste67] (see also [Ste70], Chapter VI) Stein provided an alternative approach that produced a single operator capable of extending all Sobolev spaces simultaneously.

Theorem 1.3.2 (Stein). *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain. There is a bounded linear extension operator $\mathcal{E}_{\mathcal{S}}$ such that for any $k \in \mathbb{N}$ and $1 \leq p \leq \infty$*

$$\mathcal{E}_{\mathcal{S}} : W^{k,p}(\Omega) \longrightarrow W^{k,p}(\mathbb{R}^n). \tag{1.6}$$

with bound depending on n, k, p and the constants of the Lipschitz domain.

The techniques used by Stein were quite different to those of Calderón, though they were also restricted to the case of Lipschitz domains. We note in particular that the operator no longer extends functions from $W_0^{k,p}(\Omega)$ to be zero outside Ω. Various features of Stein's method will be discussed in Section 2.3.

In [Jon81], Jones proved that Sobolev functions can be extended on locally uniform domains.

Theorem 1.3.3 (Jones). *Let $\Omega \subset \mathbb{R}^n$ be an (ϵ, δ) locally uniform domain. For each fixed $k \in \mathbb{N}$ there is a bounded linear extension operator such that for all $1 \leq p \leq \infty$*

$$\mathcal{E}^k_{\mathcal{J}} : W^{k,p}(\Omega) \longrightarrow W^{k,p}(\mathbb{R}^n) \tag{1.7}$$

with a bound depending on n, ϵ, δ, k and p.

This marked a dramatic expansion in the class of domains for which extension operators could be constructed. (We recall, for example, that a locally uniform domain may have boundary of dimension any number in $[n-1, n)$, while Lipschitz domains are bounded by locally rectifiable curves.) Moreover it provided a precise characterization of the bounded and finitely-connected extension domains in $\mathbb{R}^2$.

Theorem 1.3.4 (Jones). *If $\Omega \subset \mathbb{R}^2$ is bounded and finitely connected then the following are equivalent* *(i) There are extension operators* $\mathcal{E}^k_{\mathcal{J}}$ *as in Theorem 1.3.3.*

(ii) Ω *is an* (ϵ, ∞) *locally uniform domain.*

(iii) $\partial\Omega$ *consists of a finite number of points and quasicircles.*

Theorem 1.3.3 produces an infinite collection of extension operators $\mathcal{E}^k_{\mathcal{J}}$, one for each $k \in \mathbb{N}$. These operators are not defined on spaces with lower degrees of smoothness, nor do they operators extend functions from $W_0^{k,p}(\Omega)$ to be zero outside Ω.

Other Results

The reader will no doubt have noticed several natural questions which were not addressed in the works we have cited thus far. For example one might ask whether there are operators extending functions from $W_0^{k,p}(\Omega)$ to be zero outside Ω for a locally uniform domain Ω, whether there are analogues of the above operators for more general function spaces than the Sobolev spaces, or to what extent Theorem 1.3.4 has analogues in higher dimensions. These questions have been studied by a number of authors; among others we mention the results of Švarcman, Gol′dstein, Christ, Jonsson and Wallin, DeVore and Sharpley, Rychkov, and Koskela ([Šva78, Gn79, Chr84, JW84, DS93, Ryc99, Kos98]) which answer many of these questions. It should also be noted that Theorem 1.3.4 was preceded by a version of the same theorem for the space $W^{1,2}$, due to Gol′dstein and Vodop′anov ([GV81]), and that the question in higher dimensions has been partially addressed by Herron and Koskela ([HK92, HK91]). Recent years have seen an explosion of interest in Sobolev spaces on general metric spaces. Some extension results in this context are due to Hajłasz and Martio, Nhieu, and Harjulehto ([HM97, Nhi01, Har02]).

As none of these results are really in the direction followed in this thesis we give no further discussion of the techniques involved or the precise results obtained. Instead we identify one other problem that arises when comparing the theorems of Calderón, Stein and Jones.

Problem 1.3.5. *Given a locally uniform domain Ω, is there a single bounded linear extension operator $\mathcal{E}$ such that for all $k \in \mathbb{N}$ and $1 \leq p \leq \infty$*

$$\mathcal{E} : W^{k,p}(\Omega) \longrightarrow W^{k,p}(\mathbb{R}^n)$$

with a bound depending on n, ϵ, δ, k and p?

The purpose of the present work is to offer a solution to this problem.

Chapter 2

Constructing Extension Operators

2.1 The Main Theorem

The purpose of this thesis is to establish the following extension theorem for Sobolev spaces:

Theorem 2.1.1. *Let $\Omega \subset \mathbb{R}^n$ be an (ϵ, δ) locally uniform domain. There exists a linear operator $f \mapsto \mathcal{E}f$ such that for any $k \in \mathbb{N}$ and $1 \leq p \leq \infty$*

$$\mathcal{E} : W^{k,p}(\Omega) \longrightarrow W^{k,p}(\mathbb{R}^n) \tag{2.1}$$

$$\|\mathcal{E}f\|_{W^{k,p}(\mathbb{R}^n)} \leq c(n, \epsilon, \delta, k, p)\|f\|_{W^{k,p}(\Omega)}. \tag{2.2}$$

In this chapter we shall give the framework within which this theorem will be proved. We proceed by a method which dates back to the seminal work of Whitney [Whi34] on extensions of Lipschitz functions. Later refinements are due to Hestenes [Hes41] and Seeley [See64]. It was applied to the study of Sobolev extension operators in a manner parallel to its use here in the work of Jones [Jon80]. The method involves defining operators on a collection of Whitney cubes from the interior of $\Omega^c = \mathbb{R}^n \setminus \Omega$ and summing via a smooth partition of unity supported on the cubes, thereby reducing the extension problem for a domain to finding extensions for individual cubes that satisfy a compatibility condition from cube to cube. The relevant conditions are expressed as certain estimates for the operators corresponding to the original cubes.

The general framework just described gives a context within which it is easy to identify the essential differences between the earlier extension theorems of Calderón ([Cal61]), Stein ([Ste67], but see also [Ste70] Chapter VI, Section 3), and Jones ([Jon81]), and to provide intuition for the properties of each. We shall take some time to describe these earlier works in this manner, both because this presentation is not recorded in the literature and because it will illuminate the manner in which Theorem 2.1.1 was obtained. In particular it will be clear that our operators on Whitney cubes are related to those of Stein and that our method of proof is based on some combination of the work of Stein with that of Jones.

2.2 The Method of Whitney

The method used by Whitney to prove his celebrated extension theorem for Lipschitz functions (see [Whi34]) is the basis of the following approach to the construction of extensions of functions defined on the domain Ω.

Let $\mathcal{W}$ denote the Whitney cubes of $(\Omega^c)^o$. We begin by taking a C^∞ partition of unity $\{\Phi_Q\}$ corresponding to $\mathcal{W}$. The construction of such Φ_Q is standard, and we refer to [Ste70] Chapter VI, Section 1.3 for a proof of the following lemma.

Lemma 2.2.1. *There is a collection of functions* $\{\Phi_Q\}$ *having the properties*

- $0 \le \Phi_Q \le 1$
- *The support of* Φ_Q *lies in* $(17/16)Q$.
- $\sum_j \Phi_Q \equiv 1$ *on* $(\Omega^c)^o$.

- *For all multi-indices α, every Φ_Q satisfies the estimates*

$$|D^\alpha \Phi_Q| \leq c(|\alpha|)l(Q)^{-|\alpha|} \tag{2.3}$$

Suppose we have corresponding to each cube $Q \in \mathcal{W}$ an operator $\mathcal{E}_Q$ on locally integrable functions f and giving a function $\mathcal{E}_Q f(x)$ defined for all $x \in (17/16)Q$. We may then form an operator $\mathcal{E}$ by the locally finite sum

$$\mathcal{E}f = \sum_{Q\in\mathcal{W}} \Phi_Q \mathcal{E}_Q f \tag{2.4}$$

$$= \mathcal{E}_{Q'} f + \sum_{Q\in\mathcal{W}} (\mathcal{E}_Q f - \mathcal{E}_{Q'} f)\Phi_Q \tag{2.5}$$

where we use (2.5) to emphasize the behavior on a specific cube Q'. If each of the $\mathcal{E}_Q f$ has weak derivatives of the appropriate orders we may then differentiate to obtain

$$D^\alpha \mathcal{E} f = D^\alpha \mathcal{E}_{Q'} f + \sum_{Q\in\mathcal{W}} \sum_{0\leq\beta\leq\alpha} D^\beta(\mathcal{E}_Q f - \mathcal{E}_{Q'} f)\, D^{\alpha-\beta}\Phi_Q$$

and together with (2.3) we obtain a bound valid on Q'

$$|D^\alpha \mathcal{E} f| \leq |D^\alpha \mathcal{E}_{Q'} f| + \sum_{\{Q:Q\cap Q'\neq\emptyset\}} \sum_{0\leq\beta\leq\alpha} c(|\alpha-\beta|)l(Q')^{-|\alpha-\beta|}|D^\beta(\mathcal{E}_Q f - \mathcal{E}_{Q'} f)| \tag{2.6}$$

though it is more useful for most of our purposes to have the equivalent L^p bound. For convenience we label the neighbors of Q' by setting $\mathcal{N}(Q') = \{Q : Q \cap Q' \neq \emptyset\}$.

$$\begin{aligned} \|D^\alpha \mathcal{E} f\|_{L^p(Q')} &\leq \|D^\alpha \mathcal{E}_{Q'} f\|_{L^p(Q')} \\ &\quad + \sum_{Q\in\mathcal{N}(Q')} \sum_{0\leq\beta\leq\alpha} c(|\alpha-\beta|)l(Q')^{-|\alpha-\beta|}\|D^\beta(\mathcal{E}_Q f - \mathcal{E}_{Q'} f)\|_{L^p(Q'\cap(17/16)Q)} \end{aligned}$$

The number of terms on the right of this expression is bounded by a constant depending on n and k. It follows (from Hölder's inequality, for example) that the p-th power of the sum is at most a constant multiple of the sum of the p-th powers, and therefore

$$\begin{aligned}
&||D^\alpha \mathcal{E} f||^p_{L^p(Q')} \\
&\quad \leq C(n,k,p)\, ||D^\alpha \mathcal{E}_{Q'} f||^p_{L^p(Q')} \\
&\qquad + C(n,k,p) \sum_{Q \in \mathcal{N}(Q')} \sum_{0 \leq \beta \leq \alpha} c(|\alpha-\beta|)^p l(Q')^{-|\alpha-\beta|p} ||D^\beta(\mathcal{E}_Q f - \mathcal{E}_{Q'} f)||^p_{L^p(Q' \cap (17/16)Q)}
\end{aligned}$$

Since

$$||D^\alpha \mathcal{E} f||^p_{L^p\left(\left(\Omega^c\right)^o\right)} = \sum_{Q' \in \mathcal{W}} ||D^\alpha \mathcal{E} f||^p_{L^p(Q')}$$

we conclude that in order to prove $\mathcal{E} f \in W^{k,p}((\Omega^c)^o)$ it is sufficient to consider the quantities

$$\sum_{Q' \in \mathcal{W}} ||D^\alpha \mathcal{E}_{Q'} f||^p_{L^p(Q')} \tag{2.7}$$

$$\sum_{Q' \in \mathcal{W}} \sum_{Q \in \mathcal{N}(Q')} \sum_{0 \leq \beta \leq \alpha} c(|\alpha-\beta|)^p l(Q')^{-|\alpha-\beta|p} ||D^\beta(\mathcal{E}_Q f - \mathcal{E}_{Q'} f)||^p_{L^p(Q' \cap (17/16)Q)} \tag{2.8}$$

where it is clear that (2.7) reflects the behavior of the individual extensions $\mathcal{E}_{Q'}$, whereas (2.8) is a condition on the compatibility of $\mathcal{E}_{Q'}$ with its neighboring operators $\mathcal{E}_Q$.

At this point we have only a framework for constructing a function and estimating its derivatives. Obviously for $\mathcal{E} f(x)$ to be an extension of f there will be more work to be done, however this is not so onerous as might be supposed. In the proof of Theorem 2.1.1 the domain Ω is locally uniform, so by Lemma 1.1.4 the boundary has measure zero and no special definition of $\mathcal{E} f$ needs to be made there. It will be necessary to verify that $\mathcal{E} f$ "matches up" correctly with f at the boundary (essentially that their $(k-1)$-th derivatives are Lipschitz there - see Section 5.4), but this will be a small matter by comparison with giving an appropriate definition of the operators $\mathcal{E}_Q$ so that (2.7) and (2.8) are valid. In practice most of the new work in this thesis relates to question of how best to define the operators $\mathcal{E}_Q$. We begin this task by studying some prior work on extension operators.

2.3 The Operators of Calderón, Stein, and Jones

The Operator of Calderón

In the remarks following Theorem 1.3.1 it was mentioned that Calderón defines his extension operator via integration against a singular kernel supported on a cone. We do not wish to pursue the original definition here, but instead note in the case the function to be extended is assumed to be smooth on $\overline{\Omega}$ there is an equivalent definition in terms of the values of f and its derivatives on $\partial\Omega$. The equivalence is established using integration by parts and is outlined both in [Cal61] and [Ste70] Chapter VI, Section 4.8. There are several advantages to using this second definition, but for our purposes the main benefit is that we recognize a slight modification of Calderón's construction that easily fits the form of the Whitney method outlined above. We will not prove the following proposition as it is included here primarily for illustrative purposes.

Proposition 2.3.1. *Fix an integer k. Let Ω be a Lipschitz domain and $\mathcal{W}_1$ be all Whitney cubes of $(\Omega^c)^o$ having size less than a constant depending on the Lipschitz constant of Ω. For each $z \in \partial\Omega$ let $P_z(x)$ be the degree $(k-1)$ Taylor polynomial of f at z. If for each $Q \in \mathcal{W}_1$ we define $\mathcal{E}_Q f(x)$ to be the average with respect to arc-length of the Taylor polynomials in $10Q \cap \partial\Omega$*

$$\mathcal{E}_Q f(x) = \fint_{10Q\cap\partial\Omega} P_z(x)\, dl(z)$$

and we otherwise define $\mathcal{E}f = 0$, then $\mathcal{E}f(x)$ defined by (2.4) is an extension equivalent to that of Calderón on $W^{k,p}(\Omega) \cap C^\infty(\overline{\Omega})$. These functions are dense in $W^{k,p}(\Omega)$ and consequently the operator extends to the whole space by continuity.

In this formulation the dependence of Calderón's operator on the degree k of the Sobolev space is particularly explicit. It is also apparent that functions from $W_0^{k,p}(\Omega)$ will be extended by the zero function. One might expect that a slight modification of this definition could be used to extend germs of functions on any closed set S supporting a suitable locally finite measure $d\mu$ via

$$\mathcal{E}_Q f(x) = \fint_{10Q\cap S} P_z(x)\, d\mu(z)$$

and that with an appropriately defined function space on S this would be an extension operator. This is indeed the case under quite general circumstances. For results of this kind on Sobolev, Lipschitz, Besov, and other spaces the reader should consult the works of Jonsson and Wallin, summarized in their monograph [JW84]. We mention in passing that for a locally uniform domain it suffices for each Q to let $d\mu_Q$ be the Frostman measure on $10Q \cap \partial\Omega$.

The Operator of Jones

In [Jon80] Jones introduced a type of reflection (akin to quasiconformal reflection) that is possible on any (ϵ, δ) locally uniform domain. If $\mathcal{W}_1$ denotes the collection of Whitney cubes of Ω having size less than a constant depending on ϵ and δ, then it is possible to assign to every $Q \in \mathcal{W}_1$ a Whitney cube Q^* from the Whitney decomposition of Ω which we call the reflection of Q. The cube Q^* satisfies

$$1 \leq \frac{l(Q^*)}{l(Q)} \leq 4 \qquad\qquad \operatorname{dist}(Q, Q^*) \leq Cl(Q)$$

The reflection is not unique, however the number of cubes Q^* that could occur as reflections of a given Q is bounded by constants depending on ϵ and δ. The number of cubes Q that can share a given reflected Q^* is similarly bounded.

Let k be fixed and consider $f \in W^{k,p}(\Omega)$. The intuition underpinning Jones' extension operator is that the behavior of the extension $\mathcal{E}_Q f$ on the Whitney cube $Q \in \mathcal{W}$ should record information about the derivatives of order up to $k-1$ on a reflected cube Q^*. To this end he takes the unique polynomial $P(Q^*)$ of degree $k-1$ which best fits f on Q^* in the sense that for all $|\alpha| \leq k-1$

$$\int_{Q^*} D^\alpha(f - P(Q^*)) = 0. \tag{2.9}$$

and defines $\mathcal{E}_Q f(x) = P(Q^*)(x)$ and $\mathcal{E} f(x)$ according to (2.4).

In Section 5.3 of Chapter 5 we will see estimates akin to those used by Jones to prove that this defines an extension operator. The main technical difficulty is in obtaining estimates like (2.8), where the local connectivity property (1.4) of a locally uniform domain plays a crucial rôle. We note that Jones' operator depends explicitly on the existence of a polynomial satisfying (2.9) and consequently is not defined on the spaces $W^{l,p}(\Omega)$ for $l < k$.

The Operator of Stein

In common with the methods used by Calderón and Jones, Stein's operator is defined in a way that respects polynomial approximations of the function f. It differs in that this is achieved using a kernel that reproduces polynomials of all degrees, so is not limited to a fixed degree k of Sobolev space $W^{k,p}$.

Stein introduces a smooth function $\psi(t)$ on $[1, \infty) \subset \mathbb{R}$ with the moments

$$\int_1^\infty t^j \psi(t)\, dt = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{if } k \in \mathbb{N} \setminus \{0\} \end{cases} \tag{2.10}$$

and having a certain slow exponential decay. We will give the proof that such a function exists in Chapter 4, Section 4.1. It is clear that convolution with $\psi(t)$ reproduces polynomials in the real variable t.

Here we will modify the definition used by Stein and give instead a presentation that is better adapted to explanation within the context of the Whitney extension method. Stein's original approach may be found in [Ste70] Chapter VI, Section 3.

Let Ω be a Special Lipschitz domain and $\tilde{\Gamma}$ be a cone with vertex at the origin and of angle such that the translates of $\tilde{\Gamma}$ to points of Ω are contained entirely in Ω. Then define $\Gamma = \tilde{\Gamma} \setminus B(0,1)$. We denote points of $\mathbb{R}^n$ as (r,ξ) where $r \in [0,\infty)$ and $\xi \in S^{n-1}$, and take $\phi(\xi) \in C^\infty(S^{n-1})$ a function with $\int_{S^{n-1}} \phi(\xi) = 1$ and such that $k(r,\xi) = \psi(r)\phi(\xi)$ is supported in Γ. We note that for all polynomials $P(x)$ on $\mathbb{R}^n$

$$\int_{\mathbb{R}^n} P(x+y)k(y)\,dy = P(x) \tag{2.11}$$

Let $\mathcal{W}$ be the Whitney cubes of $(\Omega^c)^o$. To each $Q \in \mathcal{W}$ associate the cone $x_Q + \tilde{\Gamma}$ where x_Q is the center of Q. We note that there is a constant A (depending on the Lipschitz constant of Ω) such that the part of this cone that lies more than distance $Al(Q)$ from x_Q is contained in Ω. Call this set Γ_Q. By narrowing Γ and slightly increasing A we may further assume that all points within $(17/16)l(Q)$ of points in Γ_Q are in Ω. (See Figure 2.3)

Now define the operator corresponding to Q by

$$\mathcal{E}_Q f(x) = \int_\Gamma f(x + Al(Q)y)k(y)\,dy \tag{2.12}$$

Note that our choice of Γ and A ensure that $(x + Al(Q)y) \in \Omega$ whenever $x \in (17/16)Q$ and $y \in \mathrm{Sppt}(k)$.

We state without proof the following proposition, which serves primarily as motivation for our later definition of the extension operator sought in Theorem 2.1.1.

Proposition 2.3.2. *If $\mathcal{E}_Q f(x)$ is as defined in* (2.12) *then the operator $\mathcal{E}f$ defined by* (2.4) *is*

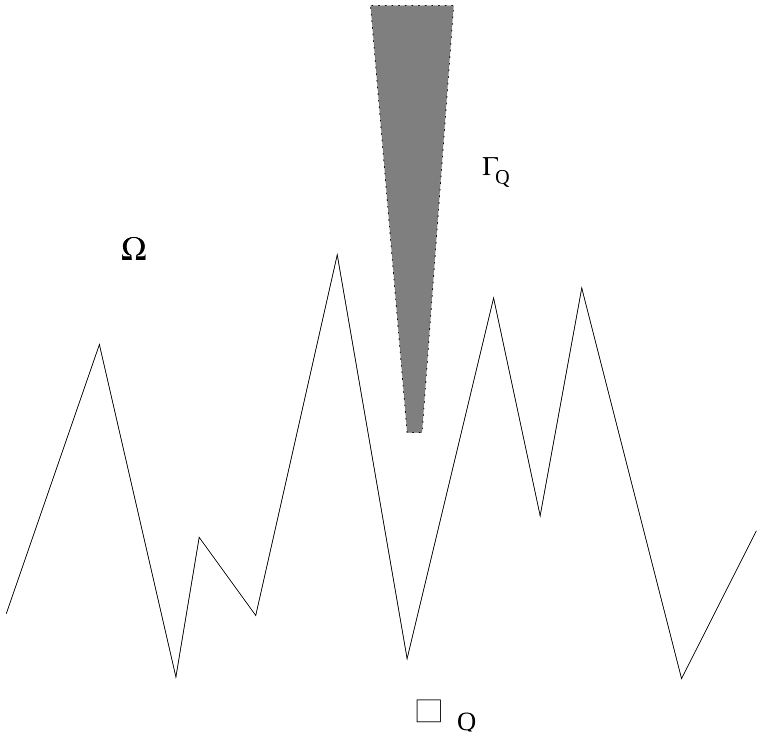

Figure 2.1: The cone Γ_Q corresponding to a cube Q.

equivalent to the extension operator constructed by Stein and referred to in Theorem 1.3.2.

The proof we give in Sections 5.3 and 5.4 of this thesis encompasses a proof that the operator just defined is in fact an extension operator on all spaces $W^{k,p}(\Omega)$. The crucial idea is the polynomial reproducing property (2.11), which will allow us to replace f by a polynomial fitted to f on part of the cone Γ_Q, in a manner similar to that seen in (2.9) of Jones' proof. The error incurred in replacing f by the polynomial will be controlled by the integral of $|\nabla^k f|$ against $|k(y)|$ on Γ_Q, so it is essential to know an estimate of the decay of the kernel $k(y)$. It is for this reason that we noted earlier that the function $\psi(t)$ has slow exponential decay. A more precise statement will be forthcoming in Chapter 3.

2.4 The Extension Operator for the Main Theorem

Our proof of Theorem 2.1.1 will be closely related to the modified version of Stein's construction that was described in Section 2.3, and in particular our definition of the operator corresponding to a cube Q will be similar to (2.12). Here we give an overview of the construction and indicate what is to come in later chapters.

Let Ω be an (ϵ, δ) locally uniform domain and $\mathcal{W}$ be the Whitney decomposition of $(\Omega^c)^o$. It should be clear from the discussion in Section 2.3 that a typical method for extending a Sobolev function $f \in W^{k,p}(\Omega)$ to a small cube $Q \in \mathcal{W}$ is to use information about polynomial approximations to f on a nearby piece of Ω. The reason the operators of Calderón and Jones depend on the smoothness k of the Sobolev space is that they are defined in terms of polynomials of fixed degree. By contrast Stein's operator makes use of a kernel that reproduces polynomials without a priori knowledge of their degree and is therefore independent of the index k. With this in mind we will define the operator for Theorem 2.1.1 using a polynomial reproducing kernel.

In the case of a Lipschitz domain it is not difficult to produce a polynomial reproducing kernel, essentially because the existence of a cone whose translates lie in Ω reduces the problem to a one dimensional question about a function with vanishing non-constant moments, as in (2.10). We will see in Chapter 4, Section 4.1 that functions of this type have been well understood for some time. For our locally uniform domain Ω the problem is substantially more difficult; most of the technical difficulties that arise in this thesis are related to this question.

The first step is to construct in Ω sets Γ_Q that correspond to the cones used in the Lipschitz case. These will only exist for small Whitney cubes, and will in general be different for each cube. They will not be cones, but will have some similarity to cones in that at a distance r from Q the set Γ_Q will contain a ball of radius comparable to r. We will think of these as *twisting cones* and will construct them in Chapter 3, Section 3.3. An example is shown in Figure 2.4.

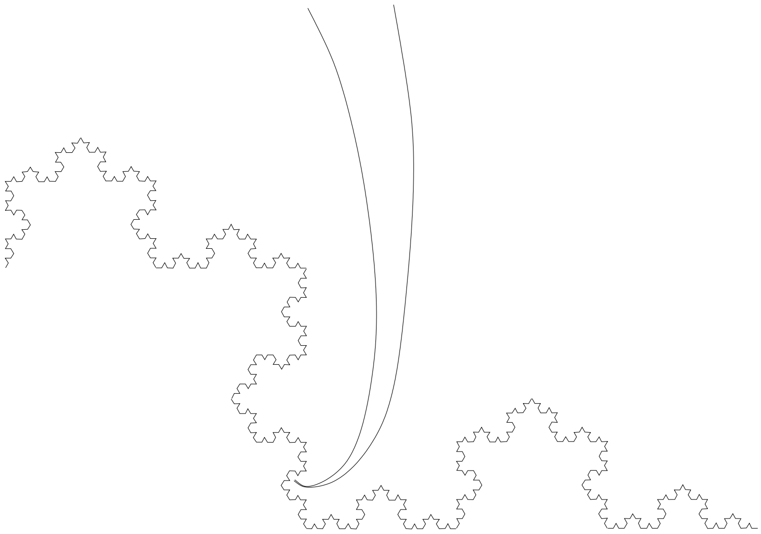

Figure 2.2: An example of a twisting cone

The crucial property of the twisting cones Γ_Q is that they are in some (measure-theoretic) sense "large enough" to support a reproducing kernel $K_Q(x)$ for polynomials. Chapter 4 is devoted to the construction of a smooth function K_Q on any twisting cone such that

$$\int_{\mathbb{R}^n} x^\alpha K_Q(x)\,dx = \begin{cases} 1 & \text{if } \alpha = (0,\dots,0) \\ 0 & \text{if } \alpha \in \mathbb{N}^n \setminus \{(0,\dots,0)\} \end{cases}$$

and therefore

$$\int_{\mathbb{R}^n} P(x+y)K_Q(y)\,dy = P(x)$$

for any polynomial $P(x)$ on $\mathbb{R}^n$. (This should be compared to (2.11).)

Once we have a kernel $K_Q(x)$ corresponding to each sufficiently small cube Q we will define $\mathcal{E}_Q f(x)$ by convolution of f with K_Q as in (2.12). For large Whitney cubes Q it will suffice to set $\mathcal{E}_Q f = 0$. The operator $\mathcal{E}$ will be the smooth sum (2.4) of the operators $\mathcal{E}_Q$.

Full details will be given in Chapter 5, Section 5.2. The bulk of Chapter 5 will be spent on obtaining estimates for terms of the form (2.7) and (2.8). An elementary argument using Proposition 4.4 of [Jon81], will then show that the resulting function is an extension.

At this point the reader should be warned that the entire proof we give for Theorem 2.1.1 is done under the additional assumption that the domain Ω has diameter at least 1. This could be avoided by renormalizing our Sobolev spaces so that the polynomials of degree $k-1$ in $W^{k,p}(\Omega)$ have norm zero when Ω has diameter less than 1, however the extra details add nothing to the proof. Nonetheless the reader should be aware that the norm of the operator on $W^{k,p}(\Omega)$ will go to infinity if the values of ϵ and δ are held fixed while the diameter of Ω goes to zero.

Chapter 3

Locally Uniform Domains

The setting for our construction of Sobolev extension operators is an (ϵ, δ) locally uniform domain Ω with diameter at least 1. Recall from (1.4) of the Preliminaries that this is the quantitative local connectivity property illustrated in Figure 3.

Definition 3.0.1. *A domain is (ϵ, δ)-locally uniform if between any pair of points x,y such that $|x - y| < \delta$ there is a rectifiable arc $\gamma(x, y) \subset \Omega$ having the properties*

$$\text{length}(\gamma) \leq \frac{|x-y|}{\epsilon} \tag{3.1}$$

$$\text{dist}(z, \partial\Omega) \geq \frac{\epsilon|z-x||z-y|}{|x-y|} \tag{3.2}$$

The geometry of Ω enters into the proof of our main result, Theorem 2.1.1, in several ways, two of which we wish to highlight here.

Recall Example 1.2.1 in which existence of an extension is obstructed by an outward cusp on $\partial\Omega$. Whenever it happens that there are arbitrarily small Whitney cubes $Q \in \mathcal{W}((\Omega^c)^o)$ such that all Whitney cubes S_j of Ω with $\text{dist}(Q, S_j) \leq Cl(Q)$ are of size $l(S_j) \ll l(Q)$ we should expect to encounter this problem. Lemma 2.4 of Jones [Jon81] shows that this cannot occur if Ω is locally uniform and $l(Q)$ sufficiently small. For the proof

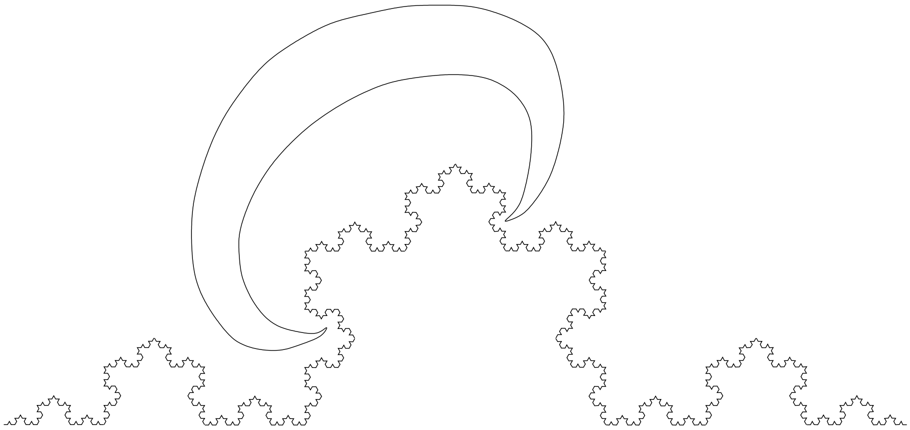

Figure 3.1: Local uniformity is a quantitative local connectivity property

of Theorem 2.1.1 we shall want somewhat more than this, requiring instead that for each $Q \in \mathcal{W}((\Omega^c)^o)$ there is a nearby set in Ω which is sufficiently large that it supports a reproducing kernel for polynomials. In Section 3.3 we produce the appropriate set which we call a twisting cone. The construction of a reproducing kernel on this type of set is the subject of Chapter 4.

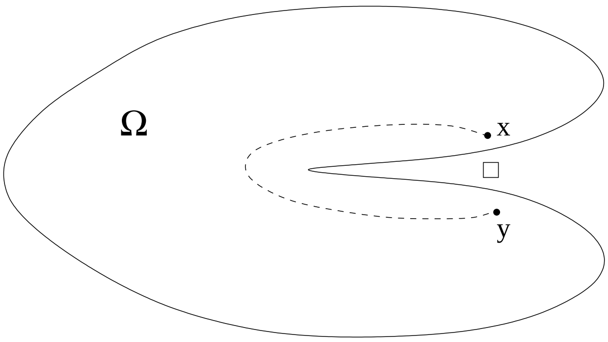

Figure 3.2: A domain with inward cusp

An equally serious obstruction to extension is illustrated in Figure 3 in which $\Omega \subset \mathbb{R}^2$ has an inward-pointing cusp. Heuristically, one can see that a function f could have small derivative on Ω but have $|f(x) - f(y)| \gg |x - y|$ for points that are close together yet separated by the cusp. This would require that the derivative of any extension be very large on Whitney cubes $Q \in \mathcal{W}((\Omega^c)^o)$ between x and y. Of course the picture is by no means a proof that extension will fail here, however this was proved by Jones in [Jon81] (see also Theorem 1.3.4 in the Preliminaries). The difficulty arises because within distance $Cl(Q)$ of $Q \in \mathcal{W}((\Omega^c)^o)$ there are large pieces of Ω which are not well connected within Ω. Conditions (3.1) and (3.2) ensure this cannot happen on a locally uniform domain by producing a tube connecting each pair $S, S' \subset \Omega$ of Whitney cubes for which the quantities $l(Q)$, $l(S)$, $l(S')$, $\mathrm{dist}(Q, S)$, and $\mathrm{dist}(Q, S')$ are all comparable. We construct such tubes in Section 3.3 and derive estimates along them in Section 3.4.

3.1 Elementary Lemmas

We will find it convenient to express a number of geometric properties of Ω in terms of Whitney cubes. As we wish to reserve the notation Q for cubes of $(\Omega^c)^o$, we shall use $S \in \mathcal{W}(\Omega)$ for cubes from the Whitney decomposition of Ω. Following Jones [Jon81] we say that two Whitney cubes *touch* if their intersection contains a face of one or both of the cubes, and that a finite sequence $\{S_1, S_2, \ldots, S_m\}$ of Whitney cubes forms a *chain* if S_j and S_{j+1} touch for $j = 1, \ldots, m$. A chain $S = S_1, \ldots S_m = S'$ is said to *connect* S and S' and to have length m.

The following lemmas are trivial (though sometimes notationally cumbersome) and included only for completeness.

Lemma 3.1.1. *Given S and S' Whitney cubes intersecting at a point x, there is a chain $\{S_j\}$ of Whitney cubes connecting S to S' and such that $x \in \cap_j S_j$.*

Proof. We may suppose without loss of generality that the point x is at the origin. Observe also that that size of the cubes are not of consequence here. We may therefore label the 2^n cubes that intersect at 0 using vectors $v = (e_1, e_2, \ldots, e_n)$ where each of the e_i is ± 1. The cube labeled $(e_1, e_2, \ldots, e_n)$ is the one that lies in the unit cube $\prod_i [0, e_i]$. Since two distinct cubes touch iff their vectors differ in a single component it is clear that a sequence $\{S_j\}$ will be a chain iff it arises from a sequence v_j of vectors in which at most one component is changed at each step. It is now obvious that starting from the v corresponding to S we may change one component at a time and (after at most n changes) obtain v' corresponding to S'. This sequence of vectors produces the desired chain. □

Lemma 3.1.2. *Given a chain $\{S_j\}$ connecting S to S' there is a chain consisting only of cubes from the original chain, connecting S to S', and having no repeated cubes.*

Proof. This is an immediate consequence of the fact that if $S_{j_1} = S_{j_2}$ with $j_1 < j_2$ then deleting the subsequence $((j_1 + 1), \ldots, j_2)$ from the chain $\{S_j\}$ produces

$$S = S_1, \ldots, S_{j_1}, S_{j_2+1}, \ldots, S_m = S'$$

which is still a chain connecting S and S'. □

Lemma 3.1.3. *If points x and y may be connected by an arc γ intersecting finitely many Whitney cubes, then the cubes $S_x \ni x$ and $S_y \ni y$ may be connected by a chain involving only cubes that intersect the arc and in which no cube is repeated.*

Proof. It suffices by the previous lemma to prove that there is a chain built from cubes intersecting the arc and connecting S_x to S_y. This may be done inductively, beginning with the trivial chain consisting only of the cube S_x. For the inductive step consider a chain of cubes taken from those intersecting γ, beginning with S_x but not including S_y. The intersection of γ with the union of the cubes in this chain contains an arc beginning at x and terminating at a point z that lies in the intersection of a cube from the chain and a cube not in the chain. It is clearly possible to extend our chain so it ends at a cube containing z, and by Lemma 3.1.1 it is then possible to choose any cube containing z but not contained in the chain and connect it to the chain using only cubes that contain the point z. All of these cubes trivially contain a point of γ. Since the number of cubes intersecting γ is finite by assumption this process must eventually join S_y to the chain, proving the lemma. □

We remark in passing that the method used in the preceding lemma produces a chain containing the arc and then trims it to remove repetitions. This may mean that the final chain does not contain the arc, however this will not be important for our purposes.

3.2 Chains between Cubes

Connecting two cubes of comparable size

Suppose that we have two Whitney cubes S and S' of Ω, separated by a distance that is comparable to the size of both cubes. Jones [Jon81] showed that in this situation the uniform domain condition implies they are connected by a chain consisting of a bounded number of cubes of controlled size. The following lemma is essentially his Lemma 2.4.

Lemma 3.2.1. *Let S and S' be Whitney cubes of Ω that have comparable sizes and separation, that is*

$$\frac{1}{C} \leq \frac{l(S)}{l(S')} \leq C, \quad \frac{1}{C} \leq \frac{|x_S - x_{S'}|}{l(S)} \leq C, \quad \frac{1}{C} \leq \frac{|x_S - x_{S'}|}{l(S')} \leq C$$

where x_S and $x_{S'}$ are the centers of S and S' respectively. Suppose also that $l(S), l(S')$ and $|x_S - x_{S'}|$ are all less than δ. Then there is a connecting chain $S = S_1, \ldots, S_m = S'$ of Whitney cubes that has finite length $m \leq C_1$, and is such that every cube S_j in the chain *satisfies*

$$\frac{\epsilon}{C_2} \leq \frac{l(S_j)}{l(S)} \leq \frac{C_2}{\epsilon} \quad \textit{and} \quad \frac{\epsilon}{C_2} \leq \frac{l(S_j)}{l(S')} \leq \frac{C_2}{\epsilon}$$

where the constants C_1 and C_2 depend only on C and n.

Proof. Consider the rectifiable curve γ connecting x_S to $x_{S'}$ and having the properties guaranteed by the local uniformity condition. Any point $z \in \gamma$ that does not lie in S or S' has $|z - x_S| \geq l(S)/2$ and $|z - x_{S'}| \geq l(S')/2$, whereupon (3.2) implies

$$\operatorname{dist}(z, \partial\Omega) \geq \frac{\epsilon l(S) l(S')}{4|x_S - x_{S'}|} \geq \frac{\epsilon |x_S - x_{S'}|^2}{4C^2 |x_S - x_{S'}|} = \frac{\epsilon |x_S - x_{S'}|}{4C^2}$$

and it follows from Lemma 1.1.2 that the Whitney cube $S(z)$ containing z has length

$$l(S(z)) \geq \frac{\epsilon |x_S - x_{S'}|}{20C^2 \sqrt{n}}$$

From this and the observation that a curve of length L intersects no more than 2^{n+1} Whitney cubes of length L we deduce that the number of cubes intersecting γ does not exceed $2^{n+1}(20C^2\sqrt{n})/\epsilon$. Moreover, a similar calculation gives an estimate on the size of the cubes involved. For z as before:

$$\begin{aligned}
\operatorname{dist}(z, \partial\Omega) &\geq \frac{\epsilon l(S) l(S')}{4|x_S - x_{S'}|} \\
&\geq \frac{\epsilon l(S) l(S')}{2C(l(S) + l(S'))} \\
&\geq \frac{\epsilon}{4C} \min\{l(S), l(S')\} \\
&\geq \frac{\epsilon}{4C^2} \max\{l(S), l(S')\}
\end{aligned}$$

and therefore

$$l(S(z)) \geq \frac{\epsilon}{20C^2 \sqrt{n}} \max\{l(S), l(S')\}$$

is the required lower bound on the size of the cubes we want for our chain. The upper bound arises even more simply from the fact that the curve has length at most $|x_S - x_{S'}|/\epsilon$ and contains x_S which has distance at most $\left(4 + \sqrt{n}/2\right) l(S)$ from $\partial\Omega$. It follows that the curve lies within $(4 + \sqrt{n}/2 + C/\epsilon)l(S)$ of $\partial\Omega$ and cannot intersect Whitney cubes of length larger than this, so that all Whitney cubes intersecting the curve γ have the size described in the conclusion of the lemma. An application of Lemma 3.1.2 now implies that this collection of cubes contains a chain of the type sought. □

Connecting a small cube to a large cube

In this context a *large* cube is one having length comparable to $\epsilon\delta/\sqrt{n}$. This is the largest size of cube which may be found all along the boundary, in the sense that any cube from Ω (or even any point of $\partial\Omega$) may be connected to a cube of this size by an arc of comparable length, and thence by a chain with known structure. This is made precise in the following lemmas.

Lemma 3.2.2. *Let $x \in \Omega$ satisfy $\operatorname{dist}(x, \partial\Omega) < \epsilon\delta/10\sqrt{n}$. Then there is a Whitney cube S of Ω with $l(S) \geq \epsilon\delta/10\sqrt{n}$, and such that x may be connected to the center x_S of S by a rectifiable curve lying within distance $2\epsilon\delta$ of $\partial\Omega$ and of length at most δ/ϵ.*

Proof. If x already lies in a Whitney cube S of side length at least $\epsilon\delta/10\sqrt{n}$ then we need only connect x to the center x_S by a straight line. It cannot lie in a larger cube as it is too close to $\partial\Omega$. Hence we assume that the Whitney cube containing x has length less than $\epsilon\delta/10\sqrt{n}$.

Since Ω is connected and of diameter at least 1 there is a point $y \in \Omega$ such that $|x-y| = \delta$. From the definition of local uniformity there is a rectifiable curve γ of length at most δ/ϵ joining x to y, and in particular containing a point z equidistant from both x and y. It is immediate that $|z - x| = |z - y| \geq \delta/2$, so at z we have by (3.2)

$$\operatorname{dist}(z, \partial\Omega) \geq \frac{\epsilon|z - x||z - y|}{|x - y|} \geq \epsilon\delta/2$$

and therefore by Lemma 1.1.2 that the Whitney cube $S' \ni z$ has length $l(S') \geq \epsilon\delta/10\sqrt{n}$.

Having exhibited a Whitney cube of length at least $\epsilon\delta/10\sqrt{n}$ on the curve from x to y it is now legitimate to take the first such cube encountered as we traverse the curve beginning at x. Call this cube S. The piece of curve connecting x to S lies entirely within cubes smaller than $\epsilon\delta/10\sqrt{n}$, hence within distance $\epsilon\delta$ of the boundary. The cube S has $l(S) \geq \epsilon\delta/10\sqrt{n}$ but must be adjacent to a cube with length smaller than that, so by (1.1) and (1.2) we have $l(S) \leq 4\epsilon\delta/10\sqrt{n}$ and it is within distance $2\epsilon\delta$ of the boundary. Moreover the curve from x to S is no longer than that from x to z, so has length at most $\delta/\epsilon - \delta/2$. We can adjoin to this curve a line segment from its endpoint on ∂S to the center x_S and have thereby connected x to x_S by a curve of total length at most

$$\delta/\epsilon - \delta/2 + \epsilon\delta/5 \leq \delta/\epsilon$$

and the proof is complete. □

Using the curves from the previous lemma it is possible to describe an aspect of the geometry near $\partial\Omega$ which will be sufficient for our construction of reproducing kernels for polynomials in Chapter 4. Corresponding to a sufficiently small Whitney cube Q of $(\Omega^c)^o$ we have a chain $\{S_j\}$ of Whitney cubes of Ω beginning at scale comparable to $l(Q)$ and separated from $l(Q)$ by distance at most $Cl(Q)$. Modulo some constant multiples the chain of cubes widens linearly, like a cone, as it connects from scale $l(Q)$ to the large scale $\epsilon\delta/10\sqrt{n}$. We think of this chain as an analogue of the cones found at boundary points of Lipschitz domains, but the chain may curve or even spiral, as shown in Figure 3.2. If it is continued to $\partial\Omega$ it may spiral infinitely, as is readily seen to be the case for the Koch snowflake domain.

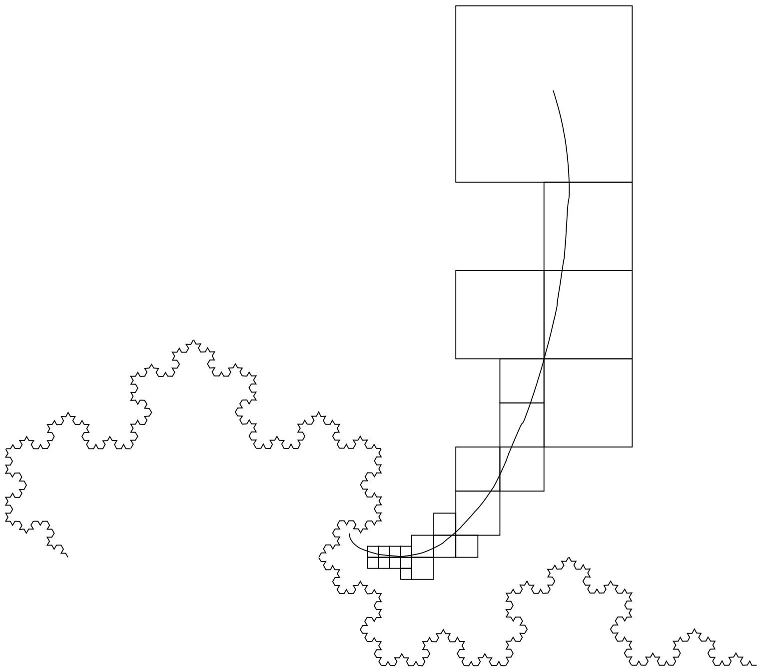

Figure 3.3: A chain of cubes in Ω

Lemma 3.2.3. *Let Q be a Whitney cube of $\mathcal{W}((\Omega^c)^o)$ with $l(Q) \leq \dfrac{\epsilon\delta}{200n}$. Then there is a Whitney cube S^* of Ω with*

$$4\sqrt{n} \leq \frac{l(S^*)}{l(Q)} \leq 16\sqrt{n} \tag{3.3}$$

$$\operatorname{dist}(Q, S^*) \leq \frac{Cn}{\epsilon} l(Q) \tag{3.4}$$

and a chain $\{S^* = S_1, S_2, \ldots, S_m = S\}$ with $l(S) \geq \epsilon\delta/10\sqrt{n}$ and having the property that

$$\frac{\epsilon}{Cn} \leq \frac{l(S_j)}{\operatorname{dist}(Q, S_j)} \leq 1 \tag{3.5}$$

where C is a constant independent of n and ϵ.

Proof. The basic properties of the Whitney decomposition (see Lemma 1.1.1) ensure that there is a point $x \in \Omega$ such that $\mathrm{dist}(x, Q) \leq 5\sqrt{n}l(Q)$. This point may be chosen as close to $\partial\Omega$ as we desire; in particular we ensure $\mathrm{dist}(x, \partial\Omega) < l(Q)$. Beginning from this point we apply Lemma 3.2.2 to obtain a curve γ connecting x to a point x_S which is the center of a Whitney cube S with $l(S) \geq \epsilon\delta/10\sqrt{n}$.

Consider the collection of cubes from $\mathcal{W}(\Omega)$ that intersect γ. By Lemma 3.1.3 we know this collection contains a chain of cubes from x to S, so we need only see that there is an appropriate starting cube on this chain and that the estimates hold. Observe that the chain contains a cube of length at most $\mathrm{dist}(x, \partial\Omega) < l(Q)$ and also a cube of length $l(S) > 16\sqrt{n}l(Q)$, hence by property (1.2) of the Whitney decomposition it certainly contains one cube of length between $4\sqrt{n}l(Q)$ and $16\sqrt{n}l(Q)$. Ordering the cubes along the chain beginning at x we call the last cube of this length S^*. Since $S^* \neq S$ we can apply the local uniformity property (3.2) to $z \in \gamma \cap S^*$ to obtain

$$
\begin{aligned}
80nl(Q) &\geq 5\sqrt{n}l(S^*) \\
&\geq \mathrm{dist}(z, \partial\Omega) \\
&\geq \frac{\epsilon|z - x||z - x_S|}{|x - x_S|} \\
&\geq \frac{\epsilon|z - x|}{2}
\end{aligned}
$$

so that $|z - x| \leq \dfrac{160n}{\epsilon}l(Q)$ and therefore $\mathrm{dist}(Q, S^*) \leq \dfrac{Cn}{\epsilon}l(Q)$

Let $\{S_j\}$ be the chain from S^* to S. By Lemma 1.1.2 we know $5\sqrt{n}l(S_j) \geq \mathrm{dist}(S_j, \partial\Omega)$. It is also clear that for any $z \in \gamma \cap S_j$

$$
\mathrm{dist}(S_j, \partial\Omega) \geq \mathrm{dist}(z, \partial\Omega) - \sqrt{n}l(S_j)
$$

therefore applying the estimate (3.2) from the locally uniform condition in the case $S_j \neq S$

$$\begin{aligned} 6\sqrt{n}l(S_j) &\geq \mathrm{dist}(z, \partial\Omega) \\ &\geq \frac{\epsilon|z-x||z-x_S|}{|x-x_S|} \\ &\geq \frac{\epsilon}{2}|z-x| \\ &\geq \frac{\epsilon}{2}(|z-x_Q| - |x_Q - x|) \\ &\geq \frac{\epsilon}{2}(\mathrm{dist}(x_Q, S_j) - 5\sqrt{n}l(Q)) \end{aligned} \tag{3.6}$$

whereupon

$$\frac{12\sqrt{n}}{\epsilon}l(S_j) \geq \mathrm{dist}(Q, S_j) - 6\sqrt{n}l(Q)$$

and using the fact that $l(S_j) \geq l(S^*) \geq 4\sqrt{n}l(Q)$ we have

$$\mathrm{dist}(Q, S_j) \leq \frac{12\sqrt{n}}{\epsilon}l(S_j) + 24nl(Q) \leq \frac{Cn}{\epsilon}l(S_j)$$

from which (3.5) follows for all cubes but S. For the cube S we can repeat the above computation for $z \in \partial S$ rather than $z \notin S$. All of the estimates are identical. □

3.3 Tubes and Twisting Cones

Construction

In order to simplify some of our proofs we will not work directly with the chains of cubes constructed in the previous section. Instead we perform an elementary construction that gives a region inside each chain on which it is easy to propagate the estimates we shall need later.

Let $\{S_j\}$ be a chain of Whitney cubes as constructed above, with no repeated cubes. For each j let a_j be the center of the cube S_j. Also let b_j be the center of the face $S_j \cap S_{j+1}$. Passing through these sequences of points in the order $a_1, b_1, a_2, \ldots, b_{m-1}, a_m$ we trace out a piecewise linear curve γ. At each point $x \in \gamma$ define a radius $s(x)$ which is $\frac{1}{2} l(S_j)$ at each x_j and $\frac{1}{2} \min\{l(S_j), l(S_{j+1})\}$ at each y_j, and between is given by the convex combination

$$s(x) = \begin{cases} (1-\lambda)\dfrac{l(S_j)}{2} + \lambda \dfrac{\min\{l(S_j), l(S_{j+1})\}}{2} & \text{if } x = (1-\lambda)x_j + \lambda y_j \\ (1-\lambda)\dfrac{\min\{l(S_j), l(S_{j+1})\}}{2} + \lambda \dfrac{l(S_{j+1})}{2} & \text{if } x = (1-\lambda)y_j + \lambda x_{j+1} \end{cases} \tag{3.7}$$

Finally, let Γ be the set of points that lie within radius $s(x)$ of some $x \in \gamma$. The result is shown in Figure 3.3.

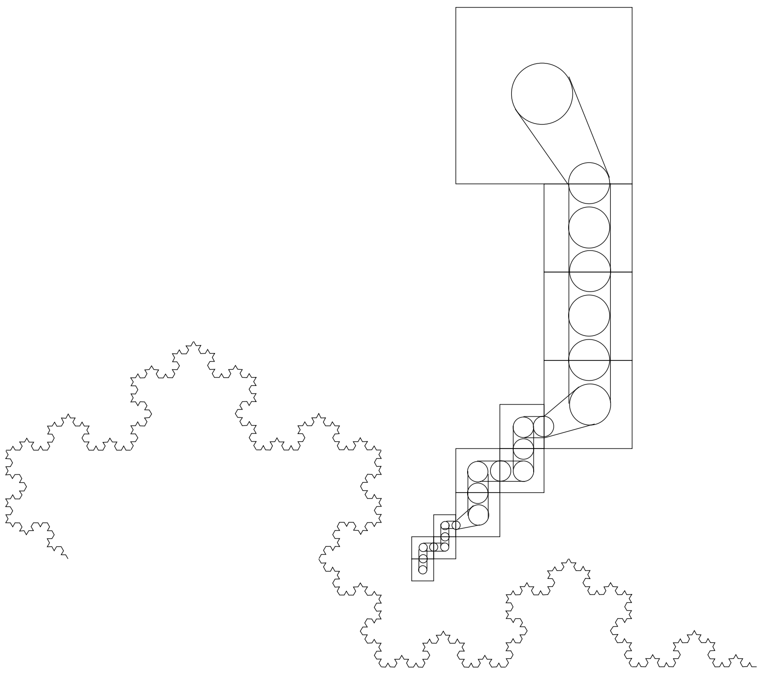

Figure 3.4: The twisting cone Γ

We record a basic property of Γ that will be useful later.

Lemma 3.3.1. *If* $y \in \Gamma \cap S_j$ *then*

$$B\left(y, 2\sqrt{n}l(Q)\right) \subset S_{j-1} \cup S_j \cup S_{j+1}$$

Proof. It is clear from the definition of Γ that all points x with

$$|x - y| \leq \frac{1}{2}\min\{l(S_{j-1}), l(S_j), l(S_{j+1})\}$$

are in $S_{j-1} \cup S_j \cup S_{j+1}$. However in the proof of Lemma 3.2.3 the smallest of the cubes S_j was S^* and had length at least $4\sqrt{n}l(Q)$ by (3.3). The lemma follows. □

If our chain is one of those described in Lemma 3.2.1 than the set Γ has radius comparable to the lengths of the cubes at its ends, with bounds depending only on ϵ, n, and the constant C in the lemma. Such Γ are called *tubes*.

In the case that the chain connects a small cube to a large cube, as in Lemma 3.2.3, we have instead that Γ is a *twisting cone*. The name describes the fact that the radius $s(x)$ is comparable to the function that grows linearly along γ and is equal to $l(S_1)$ at one end and $l(S_m)$ at the other. Like the chain that contains it, a twisting cone may contain spirals.

Counting Tubes

Motivated by the discussion at the beginning of this chapter we anticipate the need to consider for each $Q \in \mathcal{W}((\Omega^c)^o)$ the Whitney cubes $S \subset \Omega$ with $l(S) \geq C_1 l(Q)$ and $\mathrm{dist}(Q, S) \leq C_2 l(Q)$, and the tubes connecting them. The estimates below are essentially those in equations (3.1) and (3.2) of Jones [Jon81].

Fix C_1 and C_2 and let

$$\mathcal{F}(Q) = \{S_j \in \mathcal{W}(\Omega) : l(S) \geq C_1 l(Q) \text{ and } \mathrm{dist}(S, Q) \leq C_2 l(Q)\} \tag{3.8}$$

It is clear that any pair S_j, S_k from $\mathcal{F}(Q)$ satisfy the conditions of Lemma 3.2.1 so we may take a chain $\{T_l(S_j, S_k)\}$ connecting them and containing at most C_3 cubes, where $C_3 = C_3(\epsilon, n, C_1, C_2)$. As there are finitely many cubes in $\mathcal{F}(Q)$ we have

$$\left\| \sum_{S_j, S_k \in \mathcal{F}(Q)} \sum_l \Psi_{T_l(S_j,S_k)}(x) \right\|_{L^\infty} \leq C_4$$

where $\Psi_A(x)$ is the characteristic function of the set A.

Further notice that the cubes $T_l(S_j, S_k)$ all have length comparable to $l(Q)$ and that $\mathrm{dist}(Q, T_l) \leq C_5 l(Q)$. It follows that the chains arising by the above construction applied to the set $\mathcal{F}(Q')$ can only intersect those corresponding to $\mathcal{F}(Q)$ for finitely many choices of Q', and therefore that

$$\left\| \sum_{Q \in \mathcal{W}((\Omega^c)^o)} \sum_{S_j, S_k \in \mathcal{F}(Q)} \sum_l \Psi_{T_l(S_j,S_k)}(x) \right\|_{L^\infty} \leq C_6 \tag{3.9}$$

where $C_6 = C_6(\epsilon, n, C_1, C_2)$.

Counting Cones that Intersect a Cube

We record one other estimate connected to the discussion at the beginning of this chapter. We expect at some point to have operators defined by convolution against functions supported on twisting cones. Any estimates for these will need to take into account the possibility that the cones overlap, and we might therefore expect to need a bound on how many twisting cones can intersect a given Whitney cube from Ω. Unfortunately no such bound exists, and in fact most cubes will meet infinitely many twisting cones. What is true, however, is that there is a bound on the number of twisting cones Γ_Q with $\Gamma_Q \cap S \neq \emptyset$ and with Q having a fixed scale.

Suppose for each sufficiently small $Q \in \mathcal{W}((\Omega^c)^o)$ we have fixed a corresponding twisting cone Γ_Q. Fix $S \in \mathcal{W}(\Omega)$ and let $\mathcal{G}(S)$ be the set of all $Q \in \mathcal{W}((\Omega^c)^o)$ such that $\Gamma_Q \cap S \neq \emptyset$. Since the smallest cube in the chain from which Γ_Q is derived has length either $l(Q)$ or $4l(Q)$ we see that all $Q \in \mathcal{G}(S)$ have $l(Q) \leq l(S)$. Consider those Q with $2^j l(Q) = l(S)$. All of these must lie within $Cl(S)/\epsilon$ of S, otherwise the curve joining them would be of length more than $Cl(S)$ and the linear growth condition (3.5) would be violated. Within that region there are at most $(C2^j/\epsilon)^n$ candidate cubes Q, so we have shown

$$\#\{Q \in \mathcal{G}(S) : 2^j l(Q) = l(S)\} \leq C(\epsilon)2^{nj} \tag{3.10}$$

3.4 Estimation along Twisting Cones

The main purpose of introducing the notions of chains and twisting cones above was to elucidate the geometry of Ω in a fashion that allows us to estimate functions by their derivatives along chains of cubes. Essentially, what we seek is a Taylor expansion of a function f along a twisting cone. Since the functions we wish to apply this to will be Sobolev rather than smooth, the error estimates for our approximations will be of the form of generalized Poincaré inequalities. In its usual form the generalized Poincaré inequality holds for a ball, and may be written as follows

Theorem 3.4.1. *If $f \in W^{k,p}(B(0,r))$ satisfies*

$$\int_{B(0,r)} D^\alpha f = 0 \quad \textit{for all } |\alpha| \leq k-1 \tag{3.11}$$

then for all $1 \leq p \leq \infty$

$$||f||_{L^p(B(0,r))} \leq C(k)r^k ||\nabla^k||_{L^p(B(0,r))} \tag{3.12}$$

The proof of this theorem is standard. It may be found, for example, as Theorem 6.30 in [AF03], or as Lemma 1.1.11 in [Maz85].

In particular we note that from any $f \in W^{k,p}$ we may subtract the polynomial

$$P(x) = \sum_{|\alpha| \le k-1} \frac{x^\alpha}{\alpha!} \fint_B D^\alpha(\xi)\, d\xi \tag{3.13}$$

and thereby ensure $f(x) - P(x)$ satisfies (3.11). We call $P(x)$ the polynomial *fitted* to f on B.

A standard application of Theorem 3.4.1 allows estimation of the behavior of f along a sequence of overlapping balls. Under the assumption that the measure of the overlap for each pair of balls is comparable to the measure of both balls, it is possible to control the differences between successive polynomials by the L^p norm of $\nabla^k f$ on the union of the balls. Usually the comparison of two polynomials on such overlapping balls is done by noticing $\|P - \tilde{P}\|_{L^p(B_1)} \le C\|P - \tilde{P}\|_{L^p(B_2)}$. Unfortunately this approach is not optimal for our problem because the bound grows exponentially with the number of cubes traversed. For estimates along the tubes of Lemma 3.2.1 this is not an issue because the number of cubes in the chain is bounded by constants depending on the geometry of Ω, however there is no such universal bound on the number of cubes in a twisting cone. It is perhaps interesting to note that using this method gives a version of Taylor's estimate in which $\|f - P\|_{L^p}$ is bounded by d^{k+M} where d is the distance along the twisting cone and M is a constant depending on the geometry. This is in contrast to the familiar growth d^k in the classical Taylor theorem. Of course it is not possible to get exactly d^k for the situation we are considering, because the increasing width of the twisting cone implies $\|f - P\|_{L^p}$ is taken over cubes of increasing size. If we average over those cubes as in the proof of Lemma 3.4.2 below then the result is as expected.

Before we give our estimate for the behavior of f along a twisting cone Γ it will be helpful to fix some notation. Recall that Γ is centered on a piecewise linear curve γ and contained in a chain of cubes $\{S_j\}$. The vertices of γ, called a_j and b_j in Section 3.3 will here be denoted $\{z_j\}$. There is a radius $s(z)$ at each $z \in \gamma$ comparable to the size of the enclosing cube S_j. We use $B_j = B(z_j, s(z_j))$ for the balls around the vertices and $P_k(B_j; f)$ for the polynomial of degree k fitted to f on B_j.

Lemma 3.4.2. *Let $\{S_j\}$ be a chain of Whitney cubes as described in Lemma 3.2.1 or Lemma 3.2.3, and Γ be the twisting cone around γ in the chain as described in Section 3.3. Let $s(z)$ be the radius of Γ at $z \in \gamma$, write z_0 and z_m for the endpoints of γ, and $B_0 = B(z_0, s(z_0))$ and $B_m = B(z_m, s(z_m))$ for the balls around these endpoints.*

Consider $f \in W^{k,p}(\Omega)$ where $1 \le p < \infty$. If $P(x)$ is the polynomial of degree $k-1$ fitted to f on the ball B_0 then

$$\left\| f(x) - P(x) \right\|_{L^p(B_m)} \le C\,(l(S_m))^{k-1} \sum_{j=1}^{m} l(S_j) \left(\frac{l(S_m)}{l(S_j)} \right)^{n/p} \left\| \nabla^k f(y) \right\|_{L^p(S_j)} \tag{3.14}$$

while for $f \in W^{k,\infty}(\Omega)$

$$\left\| f(x) - P_Q(x) \right\|_{L^\infty(B_m)} \le C\, l(S_m)^k \left\| \nabla^k f \right\|_{L^\infty(\Omega)} \tag{3.15}$$

where $C = C(n, \epsilon, k, p)$.

Proof. Suppose $1 \le p < \infty$. We begin by examining a special case that occurs along each segment of the curve γ. Let $k = 1$ and consider the set consisting of the convex hull of the unit ball B centered at the origin and a ball of radius $(1 + \lambda)$ centered at the point a. Use $s(t) = 1 + \lambda t$ for the radius at position ta along the central axis. This is a convex set, so smooth functions are dense in the Sobolev functions (by an easy mollification argument) and it suffices to prove our estimates under the assumption that f is differentiable. For each $\xi \in B(0, 1)$ we have

$$f(a + (1 + \lambda)\xi) - f(\xi) = \int_0^1 \frac{\partial f}{\partial t}(\xi + (a + \lambda\xi)t)\, dt$$

$$= \int_0^1 \nabla f(\xi + (a + \lambda\xi)t) \cdot (a + \lambda\xi)\, dt$$

from which by Jensen's inequality and the fact $|\xi| \leq 1$

$$\begin{aligned}
\int_B \left| f(a + (1+\lambda)\xi) - f(\xi) \right|^p d\xi &\leq \int_B \int_0^1 \left| \nabla f((1+\lambda t)\xi + at) \right|^p |a + \lambda\xi|^p \, dt\, d\xi \\
&\leq (|a| + \lambda)^p \int_0^1 \int_{B(at,1)} \left| \nabla f(s(t)\xi) \right|^p d\xi\, dt \\
&\leq (|a| + \lambda)^p \int_0^1 \int_{B(at,s(t))} |\nabla f(y)|^p \frac{dy}{(s(t))^n}\, dt \qquad (3.16)
\end{aligned}$$

However the usual Poincaré theorem for $k = 1$ states

$$\int_{B(0,1)} \left| f(\xi) - ⨍_{B(0,1)} f(x)\, dx \right|^p d\xi \leq C \int_{B(0,1)} |\nabla f(\xi)|^p \, d\xi \qquad (3.17)$$

And we notice that the average of f is precisely the zero order polynomial approximation

$$P_0(B; f) = ⨍_{B(0,1)} f(x)\, dx$$

so we may combine this with (3.16) and (3.17) to obtain

$$\begin{aligned}
&\left(\int_B \left| f(a + (1+\lambda)\xi) - P_0(B; f) \right|^p d\xi \right)^{1/p} \\
&\qquad \leq C \|\nabla f\|_{L^p(B)} + \left(\int_B \left| f(a + (1+\lambda)\xi) - f(\xi) \right|^p d\xi \right)^{1/p} \\
&\qquad \leq C \|\nabla f\|_{L^p(B)} + (|a| + \lambda) \left(\int_0^1 \int_{B(at,s(t))} |\nabla f(y)|^p \frac{dy}{(s(t))^n}\, dt \right)^{1/p}
\end{aligned}$$

which by a change of variables is

$$\left(\fint_{B(a,1+\lambda)} \left|f(y) - P_0(B;f)\right|^p dy\right)^{1/p} \tag{3.18}$$

$$\leq C\|\nabla f\|_{L^p(B)} + (|a| + \lambda)\left(\int_0^1 \int_{B(at,s(t))} |\nabla f(y)|^p \frac{dy}{(s(t))^n}\, dt\right)^{1/p} \tag{3.19}$$

If we apply the Poincaré estimate (3.17) again, but this time on the ball $B' = B(a, 1+\lambda)$ we have

$$\begin{aligned}\fint_{B'} \left|f(y) - P_0(B';f)\right|^p dy &= \fint_{B'} \left|f(y) - \fint_{B'} f(x)\,dx\right|^p dy \\ &\leq C(1+\lambda)^p \fint_{B'} |\nabla f(x)|^p\,dx\end{aligned}$$

and in conjunction with (3.18) we have shown

$$\begin{aligned}\left|P_0(B';f) - P_0(B;f)\right| &\leq \left(\fint_{B'} \left|P_0(B';f) - f(y) + f(y) - P_0(B;f)\right|^p dy\right)^{1/p} \\ &\leq C(1+\lambda)\left(\fint_{B'} |\nabla f(y)|^p\,dy\right)^{1/p} + C\left(\fint_{B} |\nabla f(y)|\,dy\right)^{1/p} \\ &\quad + (|a| + \lambda)\left(\int_0^1 \fint_{B(at,s(t))} |\nabla f(y)|^p\,dy\,dt\right)^{1/p} \end{aligned} \tag{3.20}$$

We think of Γ as decomposed into a union of sets having the geometry just discussed, so $\Gamma = \cup\Gamma_l$ where Γ_l is the convex hull of $B(z_l, s(z_l))$ and $B(z_{l+1}, s(z_{l+1}))$. The estimate (3.20) applies to each Γ_l in the form

$$\begin{aligned}\left|P_0(B_l;f) - P_0(B_{l-1};f)\right| &\leq Cs(z_l)\left(\fint_{B_l} |\nabla f(y)|^p\,dy\right)^{1/p} + Cs(z_{l-1})\left(\fint_{B_{l-1}} |\nabla f(y)|\,dy\right)^{1/p} \\ &\quad + |z_l - z_{l-1}|\left(\int_{z_{l-1}}^{z_l} \fint_{B(z,s(z))} |\nabla f(y)|^p\,dy\,\frac{|dz|}{|z_l - z_{l-1}|}\right)^{1/p} \\ &\leq Cs(z_l)\left(\fint_{B_l} |\nabla f(y)|^p\,dy\right)^{1/p} + Cs(z_{l-1})\left(\fint_{B_{l-1}} |\nabla f(y)|\,dy\right)^{1/p}\end{aligned}$$

$$+ C|z_l - z_{l-1}| \left(\fint_{\Gamma_{l-1}} |\nabla f(y)|^p \, dy \right)^{1/p} \tag{3.21}$$

and we can write

$$\begin{aligned}
&\left(\fint_{B_j} \left| f(y) - P_0(B_1; f) \right|^p dy \right)^{1/p} \\
&\quad = \left(\fint_{B_j} \left| f(y) - P_0(B_j; f) + \sum_{l=1}^{j-1} (P_0(B_l; f) - P_0(B_{l-1}; f)) \right|^p dy \right)^{1/p} \\
&\quad \le \left(\fint_{B_j} \left| f(y) - P_0(B_j; f) \right|^p dy \right)^{1/p} + \sum_{l=1}^{j-1} \left| P_0(B_l; f) - P_0(B_{l-1}; f) \right| \\
&\quad \le C \sum_{l=1}^{j} s(z_l) \left(\fint_{B_l} |\nabla f(y)|^p \, dy \right)^{1/p} + C \sum_{l=1}^{j} |z_l - z_{l-1}| \left(\fint_{\Gamma_{l-1}} |\nabla f(y)|^p \, dy \right)^{1/p} \\
&\quad \le C \sum_{l=1}^{j} |z_l - z_{l-1}| \left(\fint_{\Gamma_{l-1}} |\nabla f(y)|^p \, dy \right)^{1/p}
\end{aligned} \tag{3.22}$$

where the last step uses the fact that

$$\begin{aligned}
s(z_l)^p \fint_{B_l} |\nabla f(y)|^p \, dy &= \left(\frac{s(z_l)}{|z_l - z_{l-1}|} \right)^p \frac{|\Gamma_{l-1}|}{|B_l|} |z_l - z_{l-1}|^p \fint_{\Gamma_{l-1}} |\nabla f(y)|^p \, dy \\
&\le C(p) |z_l - z_{l-1}|^p \fint_{\Gamma_{l-1}} |\nabla f(y)|^p \, dy
\end{aligned}$$

This concludes our discussion of the case $k = 1$.

Fortunately the case of general k is not dissimilar from what we have done for $k = 1$. We suppose inductively that for any smooth function g and any ball $B = B(x, s(x))$ along the segment $[z_{j-1}, z_j]$ we have

$$\begin{aligned}
&\left(\fint_{B} \left| g(y) - P_{k-2}(B_1; f) \right|^p dy \right)^{1/p} \\
&\qquad \le C \left(l(\gamma_j) \right)^{k-2} \sum_{l=1}^{j} |z_l - z_{l-1}| \left(\fint_{\Gamma_{l-1}} \left| \nabla^{k-1} g(y) \right|^p dy \right)^{1/p}
\end{aligned} \tag{3.23}$$

and we note, by a trivial computation from (3.13), that the components of $P_{k-2}(B;\nabla f)$ coincide with those of $\nabla P_{k-1}(B; f)$.

Returning to the case of a conical piece of Γ with notation as before, we follow the same method but for the function $f - P_{k-1}(B; f)$ and using the above observation about $\nabla P_{k-1}(B; f)$. Here $a = z_j - z_{j-1}$ and $\lambda + 1 = s(z_j)/s(z_{j-1})$, so that we are moving on the cone from B_{j-1} to B_j.

$$
\begin{aligned}
&⨍_{B_{j-1}} \left|(f - P_{k-1}(B; f))\,(a + \lambda\xi) - (f - P_{k-1}(B; f))\,(\xi)\right|^p d\xi \\
&\quad\le (|a| + \lambda)^p \int_0^1 ⨍_{B_{j-1}} \left|\nabla\,(f - P_{k-1}(B; f))\,(\xi + (a + \lambda\xi)t)\right|^p d\xi\, dt \\
&\quad= (|a| + \lambda)^p \int_0^1 ⨍_{B_{j-1}} \left|(\nabla f - P_{k-2}(B; \nabla f))\,(\xi + (a + \lambda\xi)t)\right|^p d\xi\, dt \\
&\quad\le C|z_j - z_{j-1}|^p \int_0^1 ⨍_{B(at,1+\lambda t)} \left|(\nabla f - P_{k-2}(B; \nabla f))\,(y)\right|^p d\xi\, dt
\end{aligned}
$$

whence by our inductive assumption applied to $g = \nabla f$, and using that $at \in [z_{j-1}, z_j]$

$$
\begin{aligned}
&\le C|z_j - z_{j-1}|^p l(\gamma_j)^{(k-2)p} \int_0^1 \left[\sum_{l=1}^{j} |z_l - z_{l-1}| \left(⨍_{\Gamma_{l-1}} |\nabla^{k-1} g(y)|^p\, dy\right)^{1/p}\right]^p dt \\
&\le Cl(\gamma_j)^{(k-2)p} |z_j - z_{j-1}|^p \left[\sum_{l=1}^{j} |z_l - z_{l-1}| \left(⨍_{\Gamma_{l-1}} |\nabla^k f(y)|^p\, dy\right)^{1/p}\right]^p
\end{aligned}
\tag{3.24}
$$

since the integrand is no longer dependent on t. We use this to write

$$
\begin{aligned}
&\left(⨍_{B_j} \left|(f - P_{k-1}(B; f))\,(y)\right|^p dy\right)^{1/p} \\
&\qquad= \left(⨍_{B_{j-1}} \left|(f - P_{k-1}(B; f))\,(a + \lambda\xi)\right|^p d\xi\right)^{1/p} \\
&\qquad\le \left(⨍_{B_{j-1}} |f(\xi) - P_{k-1}(B; f)(\xi)|^p\, d\xi\right)^{1/p}
\end{aligned}
$$

$$+ C\, l(\gamma_j)^{(k-2)}|z_j - z_{j-1}| \sum_{l=1}^{j} |z_l - z_{l-1}| \left(\fint_{\Gamma_{l-1}} |\nabla^k f(y)|^p \, dy \right)^{1/p} \tag{3.25}$$

It is clear from inductive application of (3.25) and a single use of the Poincaré inequality that

$$\begin{aligned}
&\left(\fint_{B_m} \left| (f - P_{k-1}(B; f))\,(y) \right|^p dy \right)^{1/p} \\
&\quad \leq \left(\fint_{B_1} |f(\xi) - P_{k-1}(B; f)(\xi)|^p \, d\xi \right)^{1/p} \\
&\qquad + C \sum_{j=1}^{m} l(\gamma_j)^{(k-2)}|z_j - z_{j-1}| \sum_{l=1}^{j} |z_l - z_{l-1}| \left(\fint_{\Gamma_{l-1}} |\nabla^k f(y)|^p \, dy \right)^{1/p} \\
&\quad \leq C(s(z_1))^k \left(\fint_B |\nabla^k f(y)|^p \, dy \right)^{1/p} \\
&\qquad + C \sum_{l=1}^{m} |z_l - z_{l-1}| \left(\fint_{\Gamma_{l-1}} |\nabla^k f(y)|^p \, dy \right)^{1/p} \left(\sum_{j=1}^{m} l(\gamma_j)^{(k-2)}|z_j - z_{j-1}| \right) \\
&\quad \leq C(l(\gamma_m))^{(k-1)} \sum_{l=1}^{m} |z_l - z_{l-1}| \left(\fint_{\Gamma_{l-1}} |\nabla^k f(y)|^p \, dy \right)^{1/p} \qquad (3.26)
\end{aligned}$$

Comparing this to (3.23) and using the base case $k = 1$ established in (3.22) we see that (3.26) is true for all k.

It is not difficult to pass from (3.26) to the desired estimate (3.14). The sets Γ_l are contained in cubes of the chain $\{S_j\}$. If $\Gamma_l \cap S_j \neq \emptyset$ then $|\Gamma_l|$ and $|S_j|$ are comparable and the length $|z_l - z_{l-1}|$ is is comparable to $l(S_j)$. Moreover the length $l(\gamma_j)$ is comparable to $l(S_j)$ with a constant depending on ϵ, because the length of a subarc of γ is comparable to the separation of the endpoints and we know (3.5). Multiplying both sides of (3.26) by $|B_m|^{1/p}$ and rewriting the bound in terms of $l(S_j)$ we have

$$\left\| f - P_{k-1}(B; f) \right\|_{L^p(B_m)} \leq C\, (l(S_m))^{k-1} \sum_{j=1}^{m} l(S_j) \left(\frac{l(S_m)}{l(S_j)} \right)^{n/p} \left\| \nabla^k f(y) \right\|_{L^p(S_j)}$$

This concludes the proof for the case $1 \leq p < \infty$.

When $p = \infty$ the argument is considerably simpler. We use a well known consequence of the Sobolev Embedding Theorem, namely that $f \in W^{k,p}(\Omega)$ has a representative for which $\nabla^{k-1} f$ is Lipschitz on balls contained in Ω, with Lipschitz norm $\|\nabla^k f\|_{L^\infty(\Omega)}$. Integrating $\nabla^k f$ along a rectifiable curve will then give bounds for lower order derivatives as is usual in Taylor's Theorem. As the uniform domain condition ensures that any x and y with $|x - y| < \delta$ are joined by a large number of rectifiable curves of length not exceeding $C(\epsilon)|x - y|$, we conclude immediately that

$$\left|(f(x) - P_Q(x)) - (f(y) - P_Q(y))\right| \leq C(\epsilon, k)|x - y|^k \|\nabla^k f\|_{L^\infty(\Omega)}$$

This implies both that $\left|f(x) - P_Q(x)\right|$ is bounded by $C\|\nabla^k f\|\, l(S_0)^k$ on B_0 and that $\left|f(x) - f(y)\right|$ is bounded by $C\|\nabla^k f\|\, l(S_m)^k$ for $x \in B_0$ and $y \in B_m$, so (3.15) follows and the lemma is proven. $\square$

Chapter 4

Moments and Kernels

In this Chapter we construct the reproducing kernels needed to define the operators $\mathcal{E}_Q$ for each Whitney cube Q from $(\Omega^c)^o$. The critical feature of the kernel corresponding to Q is that it should be supported on a set that is in some sense "near" to Q, yet we have only a limited amount of control on the geometry of such sets. Nonetheless we will see that the twisting cones constructed in Section 3.3 are large enough to support reproducing kernels for polynomials. Our main result is

Theorem 4.0.3. *Let $R > 0$ and $\eta < 1$ be fixed constants (which may be thought of as the initial radius and the angle of a twisting cone). If $\Gamma \subset \mathbb{R}^n$ has the property that for every $r \geq R$ there is x with $|x| = r$ and*

$$B(x, \eta|x|) \subset \Gamma \tag{4.1}$$

then there is a smooth function $K(x)$ supported on Γ and with the properties

$$\int_{\mathbb{R}^n} x^\alpha K(x)\, dx = \begin{cases} 1 & \text{if} \quad \alpha = (0, \dots, 0) \\ 0 & \text{if} \quad \alpha \in \mathbb{N}^n \setminus \{(0, \dots, 0)\} \end{cases} \tag{4.2}$$

$$\left|K(x)\right| \leq \frac{C_1}{|x|^{n-1}} \exp\left[-\left(\frac{1}{2}\log\frac{|x|}{C_2}\right)^{1/2} \exp\left(\frac{1}{2}\log\frac{|x|}{C_2}\right)^{1/2}\right] \tag{4.3}$$

where $C_1 = C_1(n, \eta, R)$ *and* $C_2 = C_2(n, \eta, R)$.

We prove this theorem using a lemma which describes the desired geometry of Γ in rather more detail.

Lemma 4.0.4. *For fixed constants* T *and* j_0 *let* $\{r_j\}$ *be the sequence*

$$r_j = T \exp\left[2\log^2(j + j_0)\right]$$

Fix also a constant t*, and suppose that* $\Gamma \subset \mathbb{R}^n$ *has the property that for each* j *there is a set* $\Xi_j \subset S^{n-1}$ *of the form*

$$\Xi_j = S^{n-1} \cap B(\xi_0, t) \tag{4.4}$$

with

$$\left\{x : r_j \leq |x| \leq r_{j+1} \text{ and } \quad \frac{x}{|x|} \in \Xi_j\right\} \subset \left(\Gamma \cap \{x : r_j \leq |x| \leq r_{j+1}\}\right) \tag{4.5}$$

then there is a smooth function $K(x)$ *supported on* Γ *which has the property* (4.2) *and satisfies the estimate* (4.3) *with constants* $C_1 = C_1(n, t, j_0, T)$ *and* $C_2 = C_2(n, t, j_0, T)$.

Proof. We prove the lemma implies the theorem by showing that the assumption (4.1) implies there are values of j_0 and T depending on R and η such that the geometry of Γ is as in Lemma 4.0.4 with $t = \eta/2$.

If we can be certain $r_0 \geq R$, then irrespective of the specific values of j_0 and T, the condition (4.1) ensures that for any j there is x_j with $|x_j| = (r_j + r_{j+1})/2$ and

$$B\left(x_j, \frac{\eta(r_j + r_{j+1})}{2}\right) \subset \Gamma$$

which in turn implies that for all $r \in [r_j, r_{j+1}]$ there are slightly smaller balls at radius r that also lie in Γ. To be concrete:

$$B\left(r\frac{x}{|x|}, \frac{1}{2}\left(\eta^2(r_j + r_{j+1})^2 - (r_{j+1} - r_j)^2\right)^{1/2}\right) \subset B\left(x_j, \frac{\eta(r_j + r_{j+1}))}{2}\right) \subset \Gamma$$

It follows immediately that we may take

$$\Xi_j = S^{n-1} \cap B\left(\frac{x}{|x|}, \frac{1}{2r_{j+1}}\left(\eta^2(r_j + r_{j+1})^2 - (r_{j+1} - r_j)^2\right)^{1/2}\right)$$

and have condition (4.5) with $t = \eta/2$, providing only that for all j

$$\frac{1}{2r_{j+1}}\left(\eta^2(r_j + r_{j+1})^2 - (r_{j+1} - r_j)^2\right)^{1/2} > \frac{\eta}{2}$$

which is the same as

$$\eta^2(r_j + r_{j+1})^2 - (r_{j+1} - r_j)^2 \geq \eta^2 r_{j+1}^2 \tag{4.6}$$

Now if $(2 - \eta)r_{j+1} \leq (2 + \eta)r_j$ we obtain

$$\begin{aligned} \eta(r_{j+1} + r_j) &\geq 2(r_{j+1} - r_j) \\ \eta^2(r_{j+1} + r_j)^2 &\geq 4(r_{j+1} - r_j)^2 \end{aligned}$$

and therefore

$$\begin{aligned} \eta^2(r_j + r_{j+1})^2 - (r_{j+1} - r_j)^2 &\geq \frac{3}{4}\eta^2(r_j + r_{j+1})^2 \\ &\geq \frac{3}{4}\left(\frac{4}{2 + \eta}\right)^2 \eta^2 r_{j+1}^2 \\ &\geq \frac{4}{3}\eta^2 r_{j+1}^2 \end{aligned}$$

so a sufficient condition for (4.6) is

$$\frac{r_{j+1}}{r_j} \leq \frac{2+\eta}{2-\eta}$$

from which it suffices that

$$\exp\left[2\log^2(j+j_0+1) - 2\log^2(j+j_0)\right] \leq \frac{2+\eta}{2-\eta} \tag{4.7}$$

We pause to notice that the derivative of $(\log^2(x+1) - \log^2 x)$ is

$$\frac{2\log(x+1)}{x+1} - \frac{2\log x}{x}$$

so that $(\log^2(x+1) - \log^2 x)$ is decreasing for $x > 1$. Moreover

$$\begin{aligned}
\log^2(x+1) - \log^2 x &= (\log(x+1) + \log x)(\log(x+1) - \log x) \\
&= \log x(x+1)\log\left(1 + \frac{1}{x}\right) \\
&\leq \frac{1}{x}\log x(x+1)
\end{aligned}$$

from which we conclude the limit of $(\log^2(x+1) - \log^2 x)$ as $x \to \infty$ is zero.

It follows that

$$\begin{aligned}
\exp\left[2\log^2(j+j_0+1) - 2\log^2(j+j_0)\right] &\leq \exp\left[2\log^2(j_0+1) - 2\log^2(j_0)\right] \\
&\leq \frac{2+\eta}{2-\eta}
\end{aligned}$$

providing that j_0 is sufficiently large. This establishes (4.7) and therefore (4.5). All that remains of the proof is to set

$$T = R\exp[-2\log^2 j_0]$$

so that $r_0 = R$ and the above reasoning is valid for all r_j. □

Most of the remainder of this chapter is spent proving Lemma 4.0.4, though we first discuss a little of the history of theorems like Theorem 4.0.3. This theorem belongs in some sense to the theory of moments, but does not seem to have attracted a lot of attention in the past. Indeed the only previous results are for sets in $\mathbb{R}$, a situation in which the geometric condition reduces to a near triviality. In Section 4.2 we present an approach to this one-dimensional problem which does not appear in the literature, and which has an additional property that is useful in establishing the result in higher dimensions. Section 4.3 is devoted to the construction of certain functions on the sets $\Xi_j \subset S^{n-1}$ and in Section 4.4 we complete the construction of $K(x)$ and the proof of Lemma 4.0.4.

4.1 Historical Remarks

The problem addressed in Theorem 4.0.3 requires finding a function with specified moments, support in a given set, and controlled decay. A special case is obtained when we restrict to the the one dimensional situation and take the set to be the half line $[1, \infty) \subset \mathbb{R}$, so that we seek a function $k(x)$ with

$$\int_1^\infty x^j k(x)\, dx = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{if } j \in \mathbb{N} \setminus \{\infty\} \end{cases} \tag{4.8}$$

In this form the problem belongs to the classical theory of moments. Early questions in the study of moments focused on whether a given sequence arises as the moments of a function or distribution, usually assumed positive, on a particular set. In particular we mention the Hamburger, Stieltjes, and Hausdorff moment problems, which ask precisely this question on each of $(-\infty, \infty)$, $[0, \infty)$ and $[0, 1]$ respectively. A complementary problem is whether such a moment sequence is unique, or equivalently, whether there is a non-zero function $f(x)$ all of whose moments are zero. By setting $f(x) = xk(x)$ we see that on $[1, \infty)$ this is the same as asking for a solution to (4.8).

The observation that (4.8) is equivalent to the uniqueness problem for a moment sequence immediately provides some limitations on the properties a solution $k(x)$ could enjoy. For example the Weierstrass approximation theorem ensures that $k(x)$ cannot be compactly supported, while the density of the Laguerre polynomials on $[0, \infty)$ with the weight e^{-x} implies that $k(x)$ cannot decay like e^{-x}. Nonetheless, in his seminal works [Sti94a, Sti94b] on continued fraction and moments, Stieltjes gave the following explicit example

$$\begin{aligned}\int_0^\infty r^k \sin(2\pi \log r) e^{-\log^2 r}\, dr &= \int_0^\infty e^{-(\log r-(k-1)/2)^2} e^{(k-1)^2/4} \sin(2\pi \log r)\, \frac{dr}{r} \\ &= e^{(k-1)^2/4} \int_{-\infty}^\infty e^{-u^2} \sin\left(2\pi\left(u + \frac{k-1}{2}\right)\right) du \\ &= 0 \quad \text{for all } k \in \mathbb{N}\end{aligned}$$

since $\sin(2\pi u + \pi(k-1))$ is an odd function. We observe that the function

$$\sin(2\pi \log r) e^{-\log^2 r}$$

has slow exponential decay.

Later work established bounds on the possible decay rates for functions of this type and produced various methods for their construction. We mention for example the criterion of Carleman for determinacy of a moment sequence (see [Car26]) and the example given by Hamburger in [Ham19] of a function with zero moments. For these and other diversions the interested reader is referred to the standard texts [ST43] and [AK62]. This is a vast theory and we cannot even survey the interesting results here. Instead we present an approach using the calculus of residues which may be found in [Ste70], Chapter VI, Section 3.2.

This is the method used by Stein to give the example mentioned in (2.10) of Chapter 2.

Consider the domain $D = \mathbb{C} \setminus [1, \infty)$. The function

$$\chi(z) = \exp(e^{i3\pi/4}(z-1)^{1/4})$$

is well defined and analytic on D and has a jump discontinuity along $\partial D = [1, \infty)$. Let γ be the closed, positively oriented contour consisting of the circular arc around 0 joining $R + i\delta$ to $R - i\delta$, two line segments on $z = \pm i\delta$, and a semicircular arc radius δ around $z = 1$. Applying Cauchy's theorem we have

$$\int_\gamma z^l \chi(z)\, dz = \begin{cases} \dfrac{2\pi i}{e} & \text{if } l = -1 \\ 0 & \text{otherwise} \end{cases}$$

Notice that $\chi(z)$ has well defined limits from above and below the line $[1, \infty)$. From above it converges to $\exp(e^{i3\pi/4}(x-1)^{1/4})$ and from below to $\exp(e^{-i3\pi/4}(x-1)^{1/4})$. Since $\chi(z)$ has rapid decay we may take the limit as $R \to \infty$ then $\delta \to 0$ to find that

$$\int_1^\infty x^l \exp(e^{i3\pi/4}(x-1)^{1/4})\, dx - \int_1^\infty x^l \exp(ie^{i3\pi/4}(x-1)^{1/4})\, dx = \begin{cases} \dfrac{2\pi i}{e} & \text{if } l = -1 \\ 0 & \text{otherwise} \end{cases}$$

from which it follows that

$$k(x) = \frac{e}{\pi x} \operatorname{Im}\Big(\exp(e^{i3\pi/4}(x-1)^{1/4})\Big)$$

has the desired moment properties. It also has much faster decay than the example given by Stieltjes, since

$$|k(x)| \le C \exp\left(\frac{-(r-1)^{1/4}}{\sqrt{2}}\right) \tag{4.9}$$

It is perhaps worth commenting on the fact that the above function may be transferred to a radial line from the origin in $\mathbb{R}^n$ and will retain the same moment properties. This not only leads easily to the construction of a similar function supported on a cone with vertex at the origin, but also leads us naturally to wonder whether the elaborate geometric conditions required in Theorem 4.0.3 are truly necessary. With the information we have available thus far there seems reason to hope we might construct the desired functions simply on curves in $\mathbb{R}^n$. The following elementary example shows this is not the case.

Consider $\mathbb{R}^2$ with co-ordinates (x, y) and the line segment γ given by $y = 1$, $x \in [1, \infty)$. This is as trivial a modification of a radial line as we might imagine, yet if we seek $K(x, y)$ on γ with the moment condition (4.2) we are doomed immediately, because we have asked for both

$$\int_\gamma K(x, y) = 1 \qquad \text{and} \qquad \int_\gamma yK(x, y) = 0$$

which is incompatible with $y = 1$ on γ. A slight modification in which we ask that $K(x, y)$ be supported on the set $|y - 1| < \epsilon$, $x \in [1, \infty)$ looks to be close to extremal, in that we might expect the size of $K(x, y)$ to be in inverse proportion to ϵ. Such a set would be much smaller than the set Γ of Theorem 4.0.3, so this further suggests the conclusions of that theorem are not sharp. This is indeed the case, and it is even possible to refine the conclusion of the theorem using only the techniques we will develop during the rest of this chapter. Such sharper results do not, however, improve our understanding of the original Sobolev extension problem, so they are not included here.

Building a kernel satisfying (4.1) will occupy the remainder of this chapter. With only the geometric information in (4.1) it is a far more technical task than the construction on $[1, \infty)$. In particular none of the methods from complex variables appear to be helpful in this situation. It should be apparent from the difficulties we encountered on γ, which was simply a translation of our well-behaved radial line, that neither the classical integration tricks nor conformal mappings are compatible with our geometric constraints. Neither are the many techniques for moment problems in higher dimensions applicable on the sets we consider. For this reason our approach to proving Lemma 4.0.4 begins by re-visiting the one-dimensional case we have just discussed, this time with the goal of constructing the function $k(x)$ in a manner that allows us to break up Γ into distinct intervals in the radial direction. Using this we will proceed in Section 4.3 to deal with the angular variables on sets of the form described in (4.5). These will combine naturally to give the construction of the desired kernel in Section 4.4.

4.2 Moments on $[1, \infty)$

We work on the half-line $I = [1, \infty) \subset \mathbb{R}$. Let $\{r_j\}_{j=0}^{\infty}$ be an increasing sequence of points from I. We consider $[r_0, \infty) \subset I$ to be partitioned into the intervals $I_j = [r_j, r_{j+1})$. Our first goal is to construct smooth functions ψ_j which have a finite number of vanishing moments and which are supported on the intervals I_j. From the functions ψ_j we will then inductively construct a function Ψ satisfying (4.2). This will require knowing estimates for the higher order moments of the ψ_j.

Some Building Blocks

Consider for each $j \in \mathbb{N}$, $j \neq 0$ the function

$$\chi_j(s) = \begin{cases} C_j \exp\left(\dfrac{j}{s^2 - 1}\right) & s \in (-1, 1) \\ 0 & \text{otherwise} \end{cases}$$

where C_j is chosen so that $\int \chi_j = 1$. For $j = 0$ set $\psi_0 = \psi_1$. It is clear that these functions are C^∞ on the real line and are supported on $[-1, 1]$. We note for future reference an elementary estimate on C_j. Observe that our functions decay monotonically as we move away from the origin and therefore that

$$\begin{aligned}\int \exp\left(\frac{j}{s^2-1}\right) ds &\geq \int_{\frac{-1}{2}}^{\frac{1}{2}} \exp\left(\frac{j}{s^2-1}\right) ds \\ &\geq \exp\left(\frac{-4j}{3}\right)\end{aligned}$$

whereupon $C_j \leq e^{4j/3}$.

We use ϕ_j to denote the function obtained by translating and scaling χ_j to the interval I_j such that ϕ_j is C^∞, supported on I_j and has $\int \phi_j = 1$.

$$\phi_j(r) = \frac{2}{(r_{j+1}-r_j)}\chi_j\left(\frac{2r}{r_{j+1}-r_j} - \frac{r_{j+1}+r_j}{r_{j+1}-r_j}\right) \tag{4.10}$$

Now we make our main definition for this section. The j-th building block function, supported on the interval I_j, is

$$\psi_j(r) = \frac{(-1)^j}{j!}\left(\frac{\partial}{\partial r}\right)^j \phi_j(r) \tag{4.11}$$

This definition is related to the classical Rodrigues formula for the Legendre polynomials. As in the theory of orthogonal polynomials, its practical application comes from the ease with which we may calculate the moments $\mu_{j,k}$ of ψ_j using integration by parts. In the following computation we differentiate r^k and integrate $\psi_j(r)$ as many as j times, noticing that at each stage the boundary terms vanish because they are multiples of derivatives of ϕ_j at the endpoints of I_j.

$$\begin{aligned}\mu_{j,k} &:= \int_I r^k \psi_j(r)\, dr \\ &= (-1)^j \frac{1}{j!}\int_{I_j} r^k \left(\frac{\partial}{\partial r}\right)^j \phi_j(r)\, dr\end{aligned}$$

$$
\begin{aligned}
&= (-1)^{j-1}\frac{k}{j!}\int_{I_j} r^{k-1}\left(\frac{\partial}{\partial r}\right)^{j-1}\phi_j(r)\,dr \\
&= (-1)^{j-2}\frac{k(k-1)}{j!}\int_{I_j} r^{k-2}\left(\frac{\partial}{\partial r}\right)^{j-2}\phi_j(r)\,dr \\
&\vdots \\
&= \begin{cases} 0 & \text{if } k < j \\ 1 & \text{if } k = j \\ \binom{k}{j}\int_{I_j} r^{k-j}\phi_j(r)\,dr & \text{if } k > j \end{cases}
\end{aligned}
\tag{4.12}
$$

At times it will be useful to change variables back to the interval $[-1, 1]$, in which case we have the expressions

$$
\mu_{j,k} = \begin{cases} 0 & \text{if } k < j \\ 1 & \text{if } k = j \\ \binom{k}{j}\left(\frac{r_{j+1}-r_j}{2}\right)^{k-j}\int_{-1}^{1}\left(s+\frac{r_{j+1}+r_j}{r_{j+1}-r_j}\right)^{k-j}\chi_j(s)\,ds & \text{if } k > j \end{cases}
\tag{4.13}
$$

for the moments of ψ_j. We also record from (4.12) that

$$
|\mu_{j,k}| \leq \binom{k}{j} r_{j+1}^{k-j}
\tag{4.14}
$$

Bounds for the building blocks

As our construction will involve adding and subtracting multiples of the functions ψ_j it will be important that we know how the L^∞ norm of ψ_j depends on j.

Lemma 4.2.1. *The functions ψ_j satisfy*

$$|\psi_j(r)| \leq \left(\frac{20}{r_{j+1} - r_j}\right)^{j+1} \tag{4.15}$$

Proof. Inserting the definition (4.10) into (4.11) and noting that the change of variables is linear we have

$$\begin{aligned}\psi_j(r) &= \frac{(-1)^j}{j!} \frac{2}{(r_{j+1} - r_j)} \left(\frac{d}{dr}\right)^j \chi_j\left(\frac{2r}{r_{j+1} - r_j} - \frac{r_{j+1} + r_j}{r_{j+1} - r_j}\right) \\ &= \frac{(-1)^j}{j!} \left(\frac{2}{(r_{j+1} - r_j)}\right)^{j+1} \left(\frac{d}{ds}\right)^j \chi_j(s) \end{aligned} \tag{4.16}$$

and we see that it suffices to know a bound for the j-th derivative of χ_j.

Rewriting the definition of $\chi_j(s)$ as

$$\chi_j(s) = C_j \exp\left(\frac{j}{s^2 - 1}\right) = C_j \exp\left(\frac{j}{2(s-1)}\right) \exp\left(\frac{-j}{2(s+1)}\right) \tag{4.17}$$

we may proceed by differentiating the product to obtain

$$C_j^{-1} \left(\frac{d}{ds}\right)^j \chi_j(s) = \sum_{k=0}^{j} \binom{j}{k} \cdot \left(\frac{d}{ds}\right)^k \exp\left(\frac{j}{2(s-1)}\right) \cdot \left(\frac{d}{ds}\right)^{j-k} \exp\left(\frac{-j}{2(s+1)}\right)$$

It is elementary but tedious to obtain bounds for these derivatives. Consider the terms that arise when we expand using the Leibnitz rule

$$\begin{aligned}\left(\frac{d}{ds}\right)^k \exp\left(\frac{j}{2(s-1)}\right) &= \left(\frac{d}{ds}\right)^{k-1} \left(\frac{-j}{2(s-1)^2}\right) \exp\left(\frac{j}{2(s-1)}\right) \\ &= \cdots \end{aligned}$$

It is clear that at all stages of the computation, the terms in the expression to be differentiated are products involving $(s-1)^{-l}\exp(j/2(s-1))$. We compute

$$\frac{d}{ds}\left[\frac{1}{(s-1)^l}\exp\left(\frac{j}{2(s-1)}\right)\right] = \frac{-l}{(s-1)^{l+1}}\exp\left(\frac{j}{2(s-1)}\right) + \frac{-j}{2(s-1)^{l+2}}\exp\left(\frac{j}{2(s-1)}\right)$$

Grouping such terms according to the homogeneity l we notice that the derivative of a term with homogeneity l consists of a term of homogeneity $l+1$ with a factor $-l$ and one of homogeneity $l+2$ with a factor $-j/2$. This allows us to describe all terms that arise in computing the k-th derivative. There are a total of 2^{k-1} terms, naturally grouped by homogeneity. Indeed, the homogeneity of a term depends on the pattern of differentiations that produced it. If l of these fell on the powers of $(s-1)$ and $(k-l)$ on the exponential factor, then by the above observation the resulting term has homogeneity $2(k-l)+l = 2k-l$. There are $\binom{k-1}{l}$ terms of this homogeneity and it is easy to deduce that the coefficients of each contain a factor of $(-j/2)^{k-l}$ from differentiation of the exponentials. The coefficients obtained by differentiating the powers are harder to write down precisely, but it is easy to see that none is as large as $(2k)^l$.

Now we need to estimate the size of a term with fixed homogeneity. As there is a trivial estimate on $[-1,0]$ we look for the maximum on $[0,1)$. Observe that for a positive value of $2k-l$

$$\begin{aligned}\log\left|\frac{1}{(s-1)^{2k-l}}\exp\left(\frac{j}{2(s-1)}\right)\right| &= -(2k-l)\log(1-s) + \frac{j}{2(s-1)}\\ \frac{d}{ds}\log\left|\frac{1}{(s-1)^{2k-l}}\exp\left(\frac{j}{2(s-1)}\right)\right| &= \frac{(2k-l)}{(1-s)} - \frac{j}{2(s-1)^2}\end{aligned}$$

so that this expression has a unique critical point in $[0,1)$ at $j/2(s-1) = -(2k-l)$. It follows that we have the bound

$$\left|\frac{1}{(s-1)^{2k-l}}\exp\left(\frac{j}{2(s-1)}\right)\right| \leq \begin{cases} \left(\dfrac{2(2k-l)}{je}\right)^{2k-l} & \text{if } 2(2k-l) \geq j \\ e^{-j/2} & \text{if } 2(2k-l) < j \end{cases} \tag{4.18}$$

where these maxima occur at the critical point and at 0 respectively.

Combining the above estimates we have bounds of the type needed in (4.17) on the interval $[0,1]$. It is an unfortunate consequence of the dichotomy in (4.18) that our bounds are different for different ranges of k. The simplest is that for $k < j/4$ where the second estimate in (4.18) must be used and we have

$$\begin{aligned} \left|\left(\frac{d}{ds}\right)^k \exp\left(\frac{j}{2(s-1)}\right)\right| &\leq e^{-j/2}\sum_{l=0}^{k-1}\binom{k-1}{l}(2k)^l\left(\frac{j}{2}\right)^{k-l} \\ &\leq e^{-j/2}\left(2k+\frac{j}{2}\right)^k \\ &\leq e^{-j/2}j^k \end{aligned}$$

For the situation in which $k \geq j/2-1$ we have $2k-j/2 \geq k-1 \geq l$ and therefore the first estimate in (4.18) is used. This gives

$$\begin{aligned} \left|\left(\frac{d}{ds}\right)^k \exp\left(\frac{j}{2(s-1)}\right)\right| &\leq \sum_{l=0}^{k-1}\binom{k-1}{l}(2k)^l\left(\frac{j}{2}\right)^{k-l}\left(\frac{2(2k-l)}{je}\right)^{2k-l} \\ &\leq \left(\frac{4k}{je}\right)^k\sum_{l=0}^{k-1}\binom{k-1}{l}(2k)^l\left(\frac{j}{2}\right)^{k-l}\left(\frac{4k}{je}\right)^{k-l} \\ &= \left(\frac{4k}{je}\right)^k\sum_{l=0}^{k-1}\binom{k-1}{l}(2k)^l\left(\frac{2k}{e}\right)^{k-l} \\ &\leq \left(\frac{4k}{je}\right)^k\left(\frac{e+1}{e}\right)^k(2k)^k \\ &\leq C^k\left(\frac{k^2}{j}\right)^k \end{aligned}$$

Finally there is the case $j/4 \leq k < j/2 - 1$ which appears at first sight to require a combination of these estimates, but for which we merely use both of the above

$$\begin{aligned}\left|\left(\frac{d}{ds}\right)^k \exp\left(\frac{j}{2(s-1)}\right)\right| &\leq \left(\frac{4k}{je}\right)^k \sum_{l=0}^{2k-j/2} \binom{k-1}{l} (2k)^l \left(\frac{2k}{e}\right)^{k-l} \\ &\quad + e^{-j/2} \sum_{l=2k-j/2}^{k-1} \binom{k-1}{l} (2k)^l \left(\frac{j}{2}\right)^{k-l} \\ &\leq C^k \left(\frac{k^2}{j}\right)^k + e^{-j/2} j^k\end{aligned}$$

This final estimate is then valid for all k.

In order to finish estimating (4.17) we need to know something about the behavior of the terms involving $(s+1)$ rather than $(s-1)$. These, however are easy. The pattern of differentiation is the same as for the $(s-1)$ terms, but on $[0,1]$ all the resulting terms are bounded by $e^{-j/2}$ because negative powers of $(s+1)$ are trivially bounded by 1. We conclude by the same method as above that

$$\left|\left(\frac{d}{ds}\right)^{j-k} \exp\left(\frac{j}{2(s+1)}\right)\right| \leq e^{-j/2} j^{(k-j)}$$

and can finally put all of our calculations together to conclude that

$$\begin{aligned}C_j^{-1}\left|\left(\frac{d}{ds}\right)^j \chi_j(s)\right| &\leq \left|\sum_{k=0}^{j} \binom{j}{k} \cdot \left(\frac{d}{ds}\right)^k \exp\left(\frac{j}{2(s-1)}\right) \cdot \left(\frac{d}{ds}\right)^{j-k} \exp\left(\frac{-j}{2(s+1)}\right)\right| \\ &\leq \sum_{k=0}^{j} \binom{j}{k} e^{-j/2} j^{(k-j)} C^k \left(\frac{k^2}{j}\right)^k + \sum_{k=0}^{j} \binom{j}{k} e^{-j} j^j \\ &\leq j^j e^{-j/2} \left[\sum_{k=0}^{j} \binom{j}{k} C^k \left(\frac{k}{j}\right)^{2k} j^{2(k-j)}\right] + 2^j e^{-j} j^j \\ &\leq j^j e^{-j/2} \left[\sum_{k=0}^{j} \binom{j}{k} C^k j^{2(k-j)}\right] + 2^j e^{-j} j^j\end{aligned}$$

$$
\begin{aligned}
&\leq j^j e^{-j/2}\left(C + j^{-2}\right)^j + 2^j e^{-j} j^j \\
&\leq j^j e^{-j}\left(e^{j/2}(C+1)^j + 2^j\right)
\end{aligned}
$$

Substituting into (4.16) and using Stirling's formula to estimate $j! \geq j^j e^{-j}\sqrt{2\pi j}$ we have at last

$$
\begin{aligned}
|\psi_j(r)| &\leq \frac{C_j j^j e^{-j}}{j^j e^{-j}\sqrt{2\pi j}}\left(e^{j/2}(C+1)^j + 2^j\right)\left(\frac{2}{(r_{j+1}-r_j)}\right)^{j+1} \\
&\leq \left(\frac{c}{r_{j+1}-r_j}\right)^{j+1}
\end{aligned}
$$

where we used the estimate $C_j \leq e^{4j/3}$. It is easily verified that we can take $c = 20$. This proves the lemma. □

Construction

Our goal is to construct a function on $[1,\infty)$ that has all its moments vanish except the one of zeroth order. A natural method to attempt is to begin with ψ_0 and inductively subtract constant multiples of the functions ψ_j for $j \geq 1$ so as to cancel each moment in turn. The induction is as follows

- The function before the j-th stage of the induction is called Ψ_j. We set $\Psi_0 = \psi_0$.
- The moments of Ψ_j are $a_k^j = \int_I r^k \Psi_j(r)\,dr$. It is then clear that $a_k^0 = \mu_{0,k}$.
- The j-th stage of the induction is

$$
\Psi_{j+1} = \Psi_j - a_{j+1}^j \psi_{j+1}
$$

from which it is clear that the moments of Ψ_{j+1} are given by

$$a_k^{j+1} = a_k^j - a_{j+1}^j \mu_{j+1,k} \tag{4.19}$$

Observe that $a_{j+1}^{j+1} = 0$ because $\mu_{j+1,j+1} = 1$. Since $\mu_{l,j+1} = 0$ for all $l > j+1$ it follows that (4.19) is actually

$$a_k^{j+1} = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } 1 \le k \le j+1 \\ a_k^j - a_{j+1}^j \mu_{j+1,k} & \text{if } k > j+1 \end{cases}$$

as was intended. Each ψ_j is supported on the interval I_j and these intervals are disjoint, so it is apparent that to prove the $\Psi_j(r)$ converge all we need do is estimate the numbers a_{j+1}^j and use our estimates on the functions ψ_j. For this purpose we define a sequence $\{b_k^j\}$ by setting $b_k^0 = |a_k^0| = |\mu_{0,k}|$ and

$$b_k^{j+1} = b_k^j + b_{j+1}^j |\mu_{j+1,k}| \tag{4.20}$$

It is clear that $|a_k^0| \le b_k^0$ for all k. Assuming inductively that $|a_k^j| \le b_k^j$ we have

$$\begin{aligned} |a_k^{j+1}| &\le |a_k^j| + |a_{j,j+1}| \mu_{j+1,k} \\ &\le b_k^j + b_{j+1}^j \mu_{j+1,k} \\ &= b_k^{j+1} \end{aligned} \tag{4.21}$$

and henceforth need only consider the sequence $\{b_{j+1}^j\}$.

Estimates

The essential idea is that binomial factor in the $\mu_{j,k}$ causes terms to increase very rapidly as j and k increase (with $k > j$). This implies that at any stage of the induction the dominant terms will be from the moments of the most recently introduced ψ_j. We do not show this explicitly because the only estimates we need are those for the numbers b^j_{j+1}, however it is the underlying philosophy of what follows here.

Lemma 4.2.2. *For $j \geq 1$ and $k \geq j$, the moments $\mu_{j,k}$ satisfy*

$$\frac{\mu_{j-1,k}}{\mu_{j-1,j}\mu_{j,k}} \leq \frac{2}{k-j+1} \tag{4.22}$$

Proof. We may explicitly compute the term $\mu_{j-1,j}$ to be

$$\begin{aligned}
\mu_{j-1,j} &= \binom{j}{j-1}\left(\frac{r_j - r_{j-1}}{2}\right)\int_{-1}^{1}\left(s + \frac{r_j + r_{j-1}}{r_j - r_{j-1}}\right)\chi_{j-1}(s)\,ds \\
&= j\left(\frac{r_j - r_{j-1}}{2}\right)\left(\frac{r_j + r_{j-1}}{r_j - r_{j-1}}\right)
\end{aligned}$$

using the fact that $\chi_{j-1}(s)$ is an even function on $[-1, 1]$.

Using the symmetry of $\phi_j(r)$ around the midpoint of I_j and the fact that r^{k-j} is an increasing function we have the bound

$$\begin{aligned}
\mu_{j,k} &= \binom{k}{j}\int_{I_j} r^{k-j}\phi_j(r)\,dr \\
&\geq \binom{k}{j}\left(\frac{r_{j+1} + r_j}{2}\right)^{k-j}
\end{aligned}$$

and we make a similarly trivial estimate on $\mu_{j-1,k}$ using the upper endpoint of the interval:

$$\begin{aligned}
\mu_{j-1,k} &= \binom{k}{j-1}\int_{I_{j-1}} r^{k-j+1}\phi_{j-1}(r)\,dr \\
&\leq \binom{k}{j-1} r_j^{k-j+1}
\end{aligned}$$

Combining these we have

$$
\begin{aligned}
\frac{\mu_{j-1,k}}{\mu_{j-1,j}\mu_{j,k}} &\leq \frac{\binom{k}{j-1} r_j^{k-j+1}}{j\binom{k}{j}\left(\frac{r_j + r_{j-1}}{2}\right)\left(\frac{r_{j+1} + r_j}{2}\right)^{k-j}} \\
&= \frac{1}{k-j+1}\left(\frac{2r_j}{r_j + r_{j-1}}\right)\left(\frac{2r_j}{r_{j+1} + r_j}\right)^{k-j} \\
&\leq \frac{2}{k-j+1}
\end{aligned}
$$

□

Lemma 4.2.3. *The sequence b_{j+1}^{j} satisfies*

$$b_{j+1}^{j} \leq e^2 b_j^{j-1} |\mu_{j,j+1}|$$

and hence

$$b_{j+1}^{j} \leq e^{2j} \prod_{l=0}^{j} |\mu_{l,l+1}| \tag{4.23}$$

Proof. We expand b_k^{j+1} using only its definition in (4.20)

$$
\begin{aligned}
b_k^{j+1} &= b_k^j + b_{j+1}^j |\mu_{j+1,k}| \\
&= b_k^{j-1} + b_j^{j-1}|\mu_{j,k}| + b_{j+1}^j |\mu_{j+1,k}| \\
&\vdots \\
&= b_k^0 + b_1^0|\mu_{1,k}| + b_2^1|\mu_{2,k}| + \cdots + b_{j+1}^j|\mu_{j+1,k}| && (4.24) \\
&= |\mu_{0,k}| + b_1^0|\mu_{1,k}| + b_2^1|\mu_{2,k}| + \cdots + b_{j+1}^j|\mu_{j+1,k}| && (4.25)
\end{aligned}
$$

and see that we must deal with a sum of terms of the type $b_l^{l-1}|\mu_{l,k}|$. Again from the definition in (4.20) we have

$$b_l^{l-1}|\mu_{l,l+1}| = b_{l+1}^{l} - b_{l+1}^{l-1} \leq b_{l+1}^{l}$$

and using this in conjunction with the inequality (4.22) from the preceding lemma we obtain for $l \geq 1$

$$\begin{aligned}
b_l^{l-1}|\mu_{l,k}| &\leq b_l^{l-1}|\mu_{l,l+1}||\mu_{l+1,k}|\left(\frac{2}{k-l}\right) \\
&\leq b_{l+1}^{l}|\mu_{l+1,k}|\left(\frac{2}{k-l}\right) \\
&\vdots \qquad \text{inductively} \\
&\leq b_{j+1}^{j}|\mu_{j+1,k}|\left(\frac{2^{j-l+1}}{(k-l)(k-l-1)\cdots(k-j)}\right)
\end{aligned}$$

while for the first term in the sum (4.24) we begin directly with the estimate (4.22) and are then in the same case as before

$$\begin{aligned}
|\mu_{0,k}| &\leq \left(\frac{2}{k}\right)|\mu_{0,1}||\mu_{1,k}| \\
&= \left(\frac{2}{k}\right)b_1^0|\mu_{1,k}| \\
&\leq b_{j+1}^{j}|\mu_{j+1,k}|\left(\frac{2^{j+1}}{k(k-1)\cdots(k-j)}\right)
\end{aligned}$$

Now we need only substitute into the sum (4.24) to find (with $m = j - l$)

$$b_k^{j+1} \leq b_{j+1}^{j}|\mu_{j+1,k}|\left(1 + \sum_{m=0}^{j}\frac{2^{m+1}(k-j-1)!}{(k-j+m)!}\right)$$

and in particular

$$\begin{aligned}
b^{j+1}_{j+2} &\le b^j_{j+1}|\mu_{j+1,j+2}|\left(1+\sum_{m=0}^{j}\frac{2^{m+1}}{(m+2)!}\right)\\
&\le b^j_{j+1}|\mu_{j+1,j+2}|\left(1+\frac{1}{2}\sum_{p=1}^{j+2}\frac{2^p}{p!}\right)\\
&\le \frac{e^2+1}{2}b^j_{j+1}|\mu_{j+1,j+2}|
\end{aligned}$$

which proves the first assertion of the lemma. The second follows from this inductively:

$$\begin{aligned}
b^j_{j+1} &\le e^2 b^{j-1}_j|\mu_{j,j+1}|\\
&\le e^4 b^{j-2}_{j-1}|\mu_{j-1,j}||\mu_{j,j+1}|\\
&\vdots\\
&\le e^{2j} b^0_1|\mu_{1,2}||\mu_{2,3}|\cdots|\mu_{j,j+1}|\\
&= e^{2j}\prod_{l=0}^{j}|\mu_{l,l+1}|
\end{aligned}$$

where the last step uses that $b^0_1 = |\mu_{0,1}|$ by definition. $\square$

Properties of $\Psi(r) = \lim \Psi_j(r)$

Recall that the functions $\Psi_j(r)$ were defined inductively by

$$\Psi_0(r) = \psi_0(r) \qquad \Psi_{j+1}(r) = \Psi_j(r) - a^j_{j+1}\psi_{j+1}(r) \tag{4.26}$$

The functions $\psi_j(r)$ are defined on the disjoint intervals I_j, so it is immediate that the $\Psi_j(r)$ converge pointwise to a function $\Psi(r)$ on I. We wish to know that this limit function decays sufficiently fast that it is integrable against all polynomials, and to know that its moments are those obtained as the limits of the moments of the $\Psi_j(r)$. To this end we employ our estimates for the functions ψ_{j+1} and for their coefficients a^j_{j+1}. By (4.21) and (4.23) we have

$$|a^j_{j+1}| \leq b^j_{j+1} \leq e^{2j} \prod_{l=0}^{j} |\mu_{l,l+1}|$$

however from (4.14) we know already that

$$|\mu_{l,l+1}| \leq (l+1) r_l$$

and so

$$|a^j_{j+1}| \leq e^{2j} (j+1)! \prod_{l=0}^{j} r_l$$

Multiplying this by ψ_{j+1}, a bound for which we found in (4.15), we have

$$|a^j_{j+1}|\,|\psi_{j+1}| \leq e^{2j} (j+1)! \left(\prod_{l=0}^{j} r_l \right) \left(\frac{20}{(r_{j+2} - r_{j+1})} \right)^{j+2} \tag{4.27}$$

and we see that this depends on our choice of the sequence $\{r_j\}$.

It is not hard to discover that the rate of growth of the sequence $\{r_j\}$ determines the bounds available from (4.27). A close to optimal choice of r_j is the sequence described in Lemma 4.0.4

$$r_j = T \exp\left[2 \log^2 (j + j_0) \right] \tag{4.28}$$

for which case we record an estimate that is useful both here and in Section 4.4.

Lemma 4.2.4. *With* $\{r_j\}$ *as in* (4.28) *and* $j_0 \geq 8$ *we have*

$$j!\left(\prod_{l=0}^{j-1} r_l\right)\left(\frac{20}{(r_{j+1}-r_j)}\right)^{j+1} \tag{4.29}$$

$$\leq \exp\Big(C + 2j_0\log^2(j+j_0) - 2(j+j_0)\log(j+j_0)\Big) \tag{4.30}$$

Proof. For notational purposes it will be convenient for us to work with the logarithm of the above quantity. The relevant estimates are

$$\begin{aligned} r_{j+1} - r_j &= T\left(\exp(2\log^2(j+j_0+1)) - \exp(2\log^2(j+j_0))\right) \\ &= T\left(\exp(2\log^2(j+j_0))\right)\left(\exp(2\log^2(j+j_0+1) - 2\log^2(j+j_0))\right) \\ &\geq T\left(\exp(2\log^2(j+j_0))\right)\left(2\log^2(j+j_0+1) - 2\log^2(j+j_0)\right) \end{aligned}$$

so that

$$\begin{aligned} \log(r_{j+1}-r_j) &\geq \log T + 2\log^2(j+j_0) + \log 2 \\ &\quad + \log\left[(\log(j+j_0+1)(j+j_0))\left(\log\left(1+\frac{1}{j+j_0}\right)\right)\right] \\ &\geq \log T + 2\log^2(j+j_0) + \log 2 + \log(2\log(j+j_0)) \\ &\quad + \log\log\left(1+\frac{1}{j+j_0}\right) \\ &\geq \log T + 2\log^2(j+j_0) + \log 4 + \log\log(j+j_0) \\ &\quad + \log\left(\frac{\log 2}{j+j_0}\right) \\ &\geq \log T + 2\log^2(j+j_0) + \log\log(j+j_0) + \log(4\log 2) \\ &\quad - \log(j+j_0) \end{aligned} \tag{4.31}$$

and for the product term

$$\begin{aligned}
\sum_{0}^{j-1} \log r_l &= j \log T + 2 \sum_{0}^{j-1} \log^2(l + j_0) \\
&\leq j \log T + 2 \int_{j_0}^{j+j_0} \log^2 x \, dx \\
&= j \log T + 2(j + j_0) \log^2(j + j_0) - 4(j + j_0) \log(j + j_0) \\
&\quad + 4(j + j_0) - 2 j_0 \log^2 j_0 + 4 j_0 \log j_0 - 4 j_0
\end{aligned} \tag{4.32}$$

Combining (4.31), (4.32), and the Stirling Estimate $j! \leq c\sqrt{j}\, j^j e^{-j}$ produces

$$\begin{aligned}
&\log\left[j! \left(\prod_{l=0}^{j-1} r_l \right) \left(\frac{20}{(r_{j+1} - r_j)} \right)^{j+1} \right] \\
&\leq \log c - j + (j + 1/2) \log j + j \log T + 2(j + j_0) \log^2(j + j_0) \\
&\quad - 4(j + j_0) \log(j + j_0) + 4j - 2 j_0 \log^2 j_0 + 4 j_0 \log j_0 \\
&\quad - (j + 1) \log T - 2(j + 1) \log^2(j + j_0) - (j + 1) \log\log(j + j_0) \\
&\quad - (j + 1) \log(4 \log 2) + (j + 1) \log(j + j_0) \\
&\leq \log c + 2 j_0 \log^2(j + j_0) - 2(j + j_0) \log(j + j_0)
\end{aligned}$$

because $j_0 \geq 8 \geq e^2$. Inserting the constant c for the Stirling estimate we obtain the conclusion of the lemma with $C = \log(\sqrt{2\pi}e)$. □

The lemma applies directly to (4.27) to give

$$\begin{aligned}
\log(|a_j^{j-1}|\,|\psi_j|) &\leq 2j + 2 j_0 \log^2(j + j_0) - 2(j + j_0) \log(j + j_0) \\
&\leq -(j + j_0 + 1) \log(j + j_0 + 1)
\end{aligned}$$

for all sufficiently large j. By (4.26) and the fact that only the only non-zero ψ_l on I_j is ψ_j, this is a bound for $|\Psi(r)|$ on the interval $I_j = [r_j, r_{j+1})$. Using $\log r \leq \log T + 2\log^2(j + j_0 + 1)$ on this interval we see that

$$\log(j + j_0 + 1) \geq \left(\frac{1}{2}\log\frac{r}{T}\right)^{1/2}$$

which gives us at last that for all sufficiently large values of r

$$\log|\Psi(r)| \leq -\left(\frac{1}{2}\log\frac{r}{T}\right)^{1/2} \exp\left(\frac{1}{2}\log\frac{r}{T}\right)^{1/2} \tag{4.33}$$

This is certainly sufficiently rapid decay to ensure integrability against the polynomials, and an application of the dominated convergence theorem shows

$$\int r^k\Psi(r)\,dr = \lim_{j\to\infty}\int r^k\Psi(r)\,dr = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k = 1, 2, 3, \ldots \end{cases} \tag{4.34}$$

so that we have found a function of the desired type on $[0, \infty)$. Our construction is cruder than the complex variable method used by Stein, so it is not surprising that we have paid a price in the decay rate of $\Psi(r)$. We saw already in (4.9) that the method he used gives a decay rate like

$$\log|K(r)| \leq \frac{-(r-1)^{1/4}}{\sqrt{2}} \leq -C\exp\left(\frac{1}{4}\log r\right) \tag{4.35}$$

which is clearly better than (4.33). In compensation we have gained substantial control over the regions in which cancellation occurs for individual monomials.

4.3 Kernels on Subsets of Spheres

In addition to our collection of kernels selecting for the radial growth r^j, we need functions that can distinguish between the many monomials that have this rate of growth. For example in $\mathbb{R}^2$ we need to be able to treat x^2, xy and y^2 independently, yet all have the radial behavior r^2. We achieve this by constructing functions on a fixed subset of the sphere $S^{n-1} \subset \mathbb{R}^n$ with the property that they vanish for all monomials except the specific one desired. It will be convenient in our construction to work with angular variables rather than the restrictions of monomials to S^{n-1}, so all our results are stated with regard to these variables. We see in Section 4.4 that this is sufficient for our problem.

The construction will be carried out first for an arc Θ on the unit circle. It will then be a simple matter to extend to the case of a subset $\Xi \subset S^{n-1}$.

Functions on an Arc of S^1

Lemma 4.3.1. *Let Θ be an arc of length $|\Theta|$ in the unit circle S^1. For a fixed $J \in \mathbb{N}$ and for each $l \in \mathbb{Z}$ with $|l| < J$ there is a smooth function $G_l(\theta)$ with support in Θ such that*

$$\int_{S^1} G_l(\theta) e^{ik\theta}\, d\theta = \begin{cases} 1 & \text{if } k = l \\ 0 & \text{if } |k| \leq J \text{ and } k \neq l \end{cases} \tag{4.36}$$

and which satifies the estimate

$$|G_l(\theta)| \leq \left(\frac{C}{|\Theta|}\right)^{2J+2} \tag{4.37}$$

Proof. The Riesz representation theorem guarantees that there is a $G_l(\theta)$ which is itself a trigonometric polynomial of degree J, hence it is reasonable to begin by solving a discretized version of the problem which constructs G_l at $2J + 1$ points. This will lead easily to a construction of $G_l(\theta)$. To simplify notation we begin with the case $l = 0$; the general case will be seen to be similar.

We partition Θ by the points $\{\lambda_0, \ldots, \lambda_{2J}\}$ where λ_0 is distance $|\Theta|/(4J+2)$ from one endpoint of Θ and $\lambda_{j+1} - \lambda_j = |\Theta|/(2J+1)$. The discretized problem is then to find numbers a_j such that

$$\sum_{j=0}^{2J} a_j e^{ik\lambda_j} = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } 1 \leq |k| \leq J \end{cases} \tag{4.38}$$

Observe that the right side of this equation has a familiar interpretation from the calculus of residues which we record as

$$\frac{1}{2\pi} \int_0^{2\pi} e^{ik\theta}\, d\theta = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \neq 1 \end{cases} \tag{4.39}$$

We use Lagrange interpolation to express the integrand $e^{ik\theta}$ in terms of the points λ_j. Writing $z_j = e^{i\lambda_j}$ we define

$$P_j(z) = \prod_{k=0, k\neq j}^{2J} \frac{z - z_k}{z_j - z_k}$$

The fact that a polynomial $Q(z)$ of degree at most $2J$ is determined by its values at $2J+1$ points allows us to write

$$Q(z) = \sum_{j=0}^{2J} Q(z_j) P_j(z)$$

In order to apply this to our integrand we set $Q(z) = z^J e^{ik\theta}$, which for $|k| \leq J$ coincides with a polynomial of the appropriate degree on the unit circle, where we conclude

$$\begin{aligned} e^{ik\theta} = \frac{Q(z)}{z^J} &= \frac{1}{z^J} \sum_{j=0}^{2J} Q(z_j) P_j(z) \\ &= e^{-J\theta} \sum_{j=0}^{2J} e^{i(J+k)\lambda_j} P_j(e^{i\theta}) \end{aligned}$$

Now integrating both sides over $\theta \in [0, 2\pi]$ yields as in (4.39)

$$\sum_{j=0}^{2J} e^{i(J+k)\lambda_j} \left(\frac{1}{2\pi} \int_0^{2\pi} P_j(e^{i\theta}) e^{-J\theta} \, d\theta \right) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } 1 \le |k| \le J \end{cases}$$

whence comparison with (4.38) yields an explicit formula for the values a_j

$$a_j = \frac{e^{iJ\lambda_j}}{2\pi} \int_0^{2\pi} P_j(e^{i\theta}) e^{-J\theta} \, d\theta \tag{4.40}$$

Using the above solution to the discretized problem we may prove the lemma by translating the set of points λ_j within the interval Θ and integrating the resulting functions against a smooth cutoff. By our choice of λ_j, the points $e^{i(\lambda_j+\phi)}$ are all in Θ for $\phi \in [-|\Theta|/(4J+2), |\Theta|/(4J+2))$. Moreover all points of Θ (except one endpoint) may be uniquely described as $e^{i\lambda_j+\phi}$ for some ϕ in this interval. Using the procedure described above, but replacing the partition $\{\lambda_j\}$ by the translates $\{\lambda_j + \phi\}$, we obtain for each $\phi \in [-|\Theta|/(4J+2), |\Theta|/(4J+2))$ a set of numbers $a_j(\phi)$ such that

$$\sum_{j=0}^{2J} a_j(\phi) e^{i(\lambda_j+\phi)k} = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } 1 \le |k| \le J \end{cases}$$

If we now take a C^∞ function $\eta(\phi)$ supported on the interval $[-|\Theta|/(4J+2), |\Theta|/(4J+2)]$ and such that $\int \eta = 1$ we find that

$$\sum_{j=0}^{2J} \int_{-|\Theta|/(4J+2)}^{|\Theta|/(4J+2)} a_j(\phi) e^{i(\lambda_j+\phi)k} \eta(\phi) \, d\phi = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } 1 \le |k| \le J \end{cases} \tag{4.41}$$

This gives a natural definition of our function $G(\theta)$. Write $\theta \in \Theta$ in its unique form $\theta = \lambda_j + \phi$

as decribed above and set

$$G(\theta) = a_j(\phi)\eta(\phi) \tag{4.42}$$

It then follows from (4.41) that

$$\begin{aligned}
\int_\Theta G(\theta)e^{ik\theta}\,d\theta &= \sum_{j=0}^{2J} \int_{\lambda_j-|\Theta|/(4J+2)}^{\lambda_j+|\Theta|/(4J+2)} G(\theta)e^{ik\theta}\,d\theta \\
&= \sum_{j=0}^{2J} \int_{-|\Theta|/(4J+2)}^{|\Theta|/(4J+2)} a_j(\phi)\eta(\phi)e^{i(\lambda_j+\phi)k}\,d\phi \\
&= \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } 1 \le |k| \le J \end{cases}
\end{aligned}$$

We now have a function $G(\theta)$ which satisfies (4.36) for the case $l = 0$. As earlier mentioned, the construction is not substantially different for general l. We merely replace (4.39) with

$$\frac{1}{2\pi}\int_0^{2\pi} e^{ik\theta}\,e^{-il\theta}d\theta = \begin{cases} 1 & \text{if } k = l \\ 0 & \text{if } k \neq l \end{cases}$$

and in later instances of integration with respect to $d\theta$ we instead use $e^{-il\theta}d\theta$. In order to simplify the derivation of the estimate (4.37) we record the precise definition of $G_j(\theta)$. First note that when we construct the Lagrange interpolating polynomials for the partition $\{\lambda_j + \phi\}$ we obtain

$$\begin{aligned}
P_{j,\phi}(z) &= \prod_{k=0,k\neq j}^{2J} \frac{z - e^{i\phi}z_k}{e^{i\phi}z_j - e^{i\phi}z_k} \\
&= P_j(e^{-i\phi}z)
\end{aligned}$$

Now we have from (4.40) that

$$a_j(\phi) = \frac{e^{iJ(\lambda_j+\phi)}}{2\pi} \int_0^{2\pi} P_j(e^{i(\theta-\phi)}) e^{-(J+l)\theta}\, d\theta$$

and therefore

$$G_l(\theta) = \frac{e^{iJ(\lambda_j+\phi)}}{2\pi} \int_0^{2\pi} P_j(e^{i(\theta-\phi)}) e^{-(J+l)\theta}\, d\theta\, \eta(\phi) \tag{4.43}$$

whence

$$|G_l(\theta)| \leq \frac{\|\eta(\phi)\|_{L^\infty}}{2\pi} \int_0^{2\pi} |P_j(e^{i\lambda})|\, d\lambda \tag{4.44}$$

Since $\eta(\phi)$ is simply a smooth cutoff function on the interval $[-|\Theta|/(4J+2), |\Theta|/(4J+2)]$ it is easily seen that $|\eta(\phi)| \leq C(2J+1)/|\Theta|$ and we are reduced to estimating P_j. This too is simple, because the all of the terms in its numerator are bounded individually by 2 for z on the unit circle and the denominator is clearly largest for the case $j = J+1$ when

$$\begin{aligned}
\prod_{k=0,k\neq j}^{2J} (z_j - z_k) &= \left(\frac{|\Theta|}{4J+2}\right)^{2J+1} 1^2 \cdot 2^2 \cdot 3^2 \cdots J^2 \\
&= \left(\frac{|\Theta|}{4J+2}\right)^{2J+1} (J!)^2 \\
&\geq \left(\frac{|\Theta|}{4J+2}\right)^{2J+1} 2\pi J^{2J+1} e^{-2J} \\
&\geq 2\pi e \left(\frac{|\Theta|}{6e}\right)^{2J+1}
\end{aligned}$$

where we used that $J! > \sqrt{2\pi J} J^J e^{-J}$ and $J/(2J+1) \geq 1/3$. From these and (4.44)

$$|G_l(\theta)| \leq \frac{C(2J+1)}{4\pi^2 e|\Theta|} \left(\frac{12e}{|\Theta|}\right)^{2J+1} \leq \left(\frac{C}{|\Theta|}\right)^{2J+2}$$

thus establishing (4.37)

It remains only to see that $G_l(\theta)$ is smooth, however we have the formula (4.43) which gives $G_l(\theta)$ explicitly as a product of smooth functions on the intervals $(\lambda_j - |\Theta|/(4J + 2), \lambda_j + |\Theta|/(4J + 2))$. At the points where two of these intervals meet we see that $\eta(\phi)$ and all its derivatives are zero, therefore the same is true of $G_l(\theta)$.

□

Functions on a subset of S^{n-1}

Consider the unit sphere $S^{n-1} \subset \mathbb{R}^n$. We use the notation $\xi \in S^{n-1}$ for points $\xi = (\xi_1, \ldots, \xi_n)$ with $\sum \xi_j^2 = 1$, and $d\sigma$ for the usual $(n-1)$ dimensional measure normalized to have unit mass on S^{n-1}. We also define generalized spherical coordinates $\theta_1, \theta_2, \ldots, \theta_{n-1}$ where $\theta_j \in [0, \pi]$ for all $j < n-1$ and $\theta_{n-1} \in [0, 2\pi)$ according to the pattern:

$$\xi_j = \begin{cases} \cos\theta_1 & \text{if } j = 1 \\ \cos\theta_j \prod_{k=1}^{j-1} \sin\theta_k & \text{if } 1 < j < n \\ \prod_{k=1}^{n-1} \sin\theta_k & \text{if } j = n \end{cases} \tag{4.45}$$

and will move freely between the notation ξ and $(\theta_1, \ldots, \theta_{n-1})$ for points of S^{n-1}. Note that the Jacobian on the sphere is $\mathcal{J} = \prod_{k=1}^{n-2} \sin^{n-k-1}\theta_k$.

A subset $\Xi \subset S^{n-1}$ is called an *angular cube* if it is of the form

$$\Xi = \{(\theta_1, \ldots, \theta_{n-1}) : \theta_j \in \Theta_j\}$$

where each $\Theta_j \subset S^1$ is an arc of length $|\Theta|$. We call $|\Theta|$ the angular length of the cube Ξ.

Lemma 4.3.2. *Let Ξ be a an angular rectangle of angular length $|\Theta|$ and such that $|\mathcal{J}| \geq C_\Xi^{-1}$ on Ξ. For a fixed $J \in \mathbb{N}$ let $\alpha = (\alpha_1, \ldots, \alpha_n)$ satisfy $|\alpha_j| \leq J$ for all j. Then there is a* *smooth function H_α supported on the set Ξ with*

$$\int_{S^{n-1}} H_\alpha(\xi)\exp\left(i\sum_{j=1}^{n-1}\beta_j\theta_j\right)d\sigma(\xi) = \begin{cases} 1 & \text{if } \beta = \alpha \\ 0 & \text{if } |\beta_j| \le J \text{ for all } j \text{ and } \beta \neq \alpha \end{cases}$$

and such that H_α satisfies the estimate

$$|H_\alpha| \le C_\Xi \left(\frac{C}{|\Theta|}\right)^{2(n-1)(J+1)} \tag{4.46}$$

Proof. We use Lemma 4.3.1 to define functions $G_{\alpha_j}(\theta)$ supported on Θ_j and having

$$\int G_{\alpha_j}(\theta)e^{ik\theta}\,d\theta = \begin{cases} 1 & \text{if } k = \alpha_j \\ 0 & \text{if } |k| \le J \text{ and } k \neq \alpha_j \end{cases} \tag{4.47}$$

For each j we may use (4.37) to obtain

$$|G_{\alpha_j}| \le \left(\frac{C}{|\Theta_j|}\right)^{2(J+1)} \tag{4.48}$$

Define

$$H_\alpha = \frac{1}{\mathcal{J}}\prod_{j=1}^{n-1} G_{\alpha_j}(\theta_j)$$

By applying (4.47) we see that

$$\begin{aligned} &\int_{S^{n-1}} H_\alpha(\xi)\exp\left(i\sum_{j=1}^{n-1}\beta_j\theta_j\right)d\sigma(\xi) \\ &= \int_{S^{n-1}}\left(\prod_{j=1}^{n-1} G_{\alpha_j}(\theta_j)\right)\left(\prod_{j=1}^{n} e^{i\alpha_j\theta_j}\right)d\theta_1\cdots d\theta_{n-1} \\ &= \prod_{j=1}^{n-1}\int_0^\pi G_{\alpha_j}(\theta_j)e^{i\alpha_j\theta_j}\,d\theta_j \end{aligned}$$

$$= \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if some } \beta_j \leq J \text{ and } \alpha \neq \beta \end{cases}$$

and we can estimate the size of H_α on Ξ using the assumption $|\mathcal{J}| \geq C_\Xi^{-1}$ and (4.48) to obtain

$$\begin{aligned} |H_\alpha(\xi)| &\leq \frac{1}{\mathcal{J}} \prod_{j=1}^{n-1} \left(\frac{C}{|\Theta|} \right)^{2(J+1)} \\ &\leq C_\Xi \left(\frac{C}{\Theta|} \right)^{2(n-1)(J+1)} \end{aligned}$$

□

Our construction on an angular cube is useful because the intersection of a twisting cone as in 3.3 with a sphere around the origin contains an angular cube of some fixed size. This, and the verification that we can avoid locations where $\mathcal{J}$ is large, is the content of Lemma 4.3.3.

Lemma 4.3.3. *If $\upsilon \in S^{n-1}$ and $t < 1$ then the set $S^{n-1} \cap B(\upsilon, t)$ contains an angular cube with angular length $|\Theta| \leq C_1 t$ and on which $|\mathcal{J}| \geq C_2 t^{n-2}$. The constants C_1 and C_2 depend only upon the dimension n.*

Proof. We verify the assertion about $\mathcal{J}$ by showing that there is a constant λ such that $B(\upsilon, t)$ contains a ball $B(\tilde{\upsilon}, \lambda t)$ centered at $\tilde{\upsilon} \in S^{n-1}$ and on which $\mathcal{J}$ is appropriately bounded. Observe that such a bound holds on the set

$$\{\xi = (\xi_1, \ldots, \xi_n) \in S^{n-1} : \xi_{n-1}^2 + \xi_n^2 \geq (\lambda t)^2\}$$

because we have

$$|\mathcal{J}| = \prod_{k=1}^{n-2} |\sin\theta_k|^{n-k-1} \geq \left(\prod_{k=1}^{n-2} |\sin\theta_k|\right)^{n-2} = \left(\xi_{n-1}^2 + \xi_n^2\right)^{(n-2)/2}$$

so that $|\mathcal{J}| \geq (\lambda t)^{n-2}$.

We may restrict to the case $\upsilon \in \{\xi_{n-1}^2 + \xi_n^2 \leq (2\lambda t)^2\}$, as otherwise the ball $B(\upsilon, \lambda t)$ has the advertised property. Let $\tilde{\upsilon}$ be the point of $S^{n-1} \cap \{\xi_{n-1}^2 + \xi_n^2 = (2\lambda t)^2\}$ that is closest to υ. We claim $|\upsilon - \tilde{\upsilon}| \leq 2\sqrt{2}\lambda t$. To see this write $\upsilon = (\upsilon_1, \ldots, \upsilon_n)$ and verify that the set contains the point τ defined by

$$\tau_j = \left(\frac{1-(2\lambda t)^2}{1-\upsilon_{n-1}^2-\upsilon_n^2}\right)^{1/2} \upsilon_j \quad \text{for } j = 1, \ldots, n-2$$

$$(\tau_{n-1}, \tau_n) = \begin{cases} \dfrac{2\lambda t}{(\upsilon_{n-1}^2+\upsilon_n^2)^{1/2}}(\upsilon_{n-1}, \upsilon_n) & \text{if } \upsilon_{n-1}^2 + \upsilon_n^2 \neq 0 \\ (2\lambda t, 0) & \text{if } \upsilon_{n-1}^2 + \upsilon_n^2 = 0 \end{cases}$$

Providing $\upsilon_{n-1}^2 + \upsilon_n^2 \neq 0$ we have a bound

$$\begin{aligned} |\upsilon - \tau|^2 &= \sum_{j=1}^{n} |\upsilon_j - \tau_j|^2 \\ &= \left[1 - \frac{2\lambda t}{(\upsilon_{n-1}^2+\upsilon_n^2)^{1/2}}\right]^2 (\upsilon_{n-1}^2 + \upsilon_n^2) + \left[1 - \left(\frac{1-(2\lambda t)^2}{1-\upsilon_{n-1}^2-\upsilon_n^2}\right)^{1/2}\right]^2 \sum_{j=1}^{n-2} \upsilon_j^2 \\ &\leq (2\lambda t)^2 + (2\lambda t)^2 \sum_{j=1}^{n-2} \upsilon_j^2 \\ &\leq 2(2\lambda t)^2 \end{aligned}$$

where we used

$$\left(\frac{1-(2\lambda t)^2}{1-\upsilon_{n-1}^2-\upsilon_n^2}\right)^{1/2} \geq (1-(2\lambda t)^2)^{1/2} \geq 1 - 2\lambda t \quad \text{for} \quad 2\lambda t \leq 1 \tag{4.49}$$

The same estimate for $|\upsilon - \tau|$ is even easier in the case $\upsilon_{n-1}^2 + \upsilon_n^2 = 0$, so the claim is proven providing $2\lambda t < 1$.

We now need only observe from $|\upsilon - \tilde{\upsilon}| \leq 2\sqrt{2}\lambda t$ and the definition of $\tilde{\upsilon}$ that

$$B(\tilde{\upsilon}, (1 - 2\sqrt{2}\lambda)t) \subset B(\upsilon, t)$$

$$B(\tilde{\upsilon}, \lambda t) \subset \{\xi = (\xi_1, \ldots, \xi_n) : \xi_{n-1}^2 + \xi_n^2 \geq (\lambda t)^2\}$$

Setting $(1 - 2\sqrt{2}\lambda) = \lambda$, i.e. $\lambda = 1/(1 + 2\sqrt{2})$, we see that $2\lambda t < 1$ so (4.49) is valid, and that the ball $B(\tilde{\upsilon}, \lambda t)$ has all the properties we desired.

By the above argument it suffices to assume the ball $B(\upsilon, t)$ satisfies the bound on $|\mathcal{J}|$ and to see that it contains an angular cube. However it is clear from (4.45) that at any point $(\theta_1, \ldots, \theta_{n-1})$ on the sphere, changing θ_j by an amount ϕ moves the point by Euclidean distance less than $|\phi|$. In particular if $\upsilon = (\theta_1, \ldots, \theta_{n-1})$ then

$$\left\{(\phi_1, \ldots, \phi_{n-1}) : |\theta - \phi| \leq \frac{t}{\sqrt{n}}\right\} \subset B(\upsilon, t)$$

is an angular cube of the desired type. $\square$

As promised, we now have an appropriate function on the types of subsets of S^{n-1} that arise in the case of twisting cones. Combining Lemmas 4.3.2 and 4.3.3 we have proven

Corollary 4.3.4. *Let $\upsilon \in S^{n-1}$ and $t < 1$. Fix $J \in \mathbb{N}$ and let $\alpha = (\alpha_1, \ldots, \alpha_n)$ satisfy $|\alpha_j| \leq J$ for all j. Then the set $S^{n-1} \cap B(\upsilon, t)$ supports a smooth function H_α with*

$$\int_{S^{n-1}} H_\alpha(\xi) \exp\left(i \sum_{j=1}^{n-1} \beta_j \theta_j\right) d\sigma(\xi) = \begin{cases} 1 & \text{if } \beta = \alpha \\ 0 & \text{if } |\beta_j| \leq J \text{ for all } j \text{ and } \beta \neq \alpha \end{cases}$$

and such that H_α satisfies the estimate

$$|H_\alpha| \leq \left(\frac{C}{t}\right)^{(n-1)(2J+3)} \tag{4.50}$$

Remarks

The reader may wonder why we define complex valued functions when our eventual goal is a real valued kernel $K(x)$ with the properties listed in Theorem 4.0.3. The basic idea is that the restriction of a monomial in x to the sphere will give a polynomial in the sines and cosines of the angular variables, and this can be expressed as a polynomial in the exponential monomials $e^{i\alpha\theta}$. This (real-valued) polynomial in $e^{i\alpha\theta}$ will integrate to zero against the (complex-valued) kernel, and therefore will integrate to zero against the real part of the kernel, which will be $K(x)$. While it would be possible to deal directly with the sine and cosine functions at this point in the proof, it is notationally simpler to use the method we have been following. Nonetheless it is apparent that the above arguments, particularly for the case of the sphere S^{n-1}, have been chosen more for their brevity and simplicity than for the precision of the estimates that result. It is possible to do a more careful construction that produces somewhat better estimates on the decay of the functions H_α, and it is certainly possible to do both the construction of $G_l(\theta)$ and of $H_\alpha(\xi)$ on more general subsets of the sphere than those used here.

4.4 The Kernel on Γ

Building Blocks and Bounds

The hypotheses of Lemma 4.0.4 provide a decomposition of Γ into

$$\Gamma_j = \left\{ r_j \leq |x| \leq r_{j+1}, \frac{x}{|x|} \in \Xi_j \right\}$$

$$\Xi_j = S^{n-1} \cap B(\xi_j, t)$$

where t is independent of j. Writing $I_j = [r_j, r_{j+1})$ we associate to each radial interval I_j the function $\psi_j(r)$ of Section 4.2. For each j we then apply the result of Corollary 4.3.4 to the set $\Xi_j \subset S^{n-1}$. Setting $J = 2j + 2$ we construct, for each multi-index $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$ with all $|\alpha_l| \leq j$, smooth functions H_α supported on Xi_j and satisfying

$$\int_{S^{n-1}} H_{j,\alpha}(\xi) \exp\left(i \sum_{j=1}^{n-1} \beta_j \theta_j \right) d\sigma(\xi) = \begin{cases} 1 & \text{if } \beta = \alpha \\ 0 & \text{if } |\beta_l| \leq 2j + 2 \text{ for all } l, \text{ and } \beta \neq \alpha \end{cases} \tag{4.51}$$

as well as the estimate

$$|H_\alpha| \leq \left(\frac{C}{t} \right)^{(n-1)(4j+7)} \tag{4.52}$$

We then combine these with the radial functions $\psi_j(r)$ to define

$$F_{(j,\alpha)}(r, \xi) = \psi_j(r) H_{j,\alpha}(\xi)$$

The functions $F_{(j,\alpha)}(r, \xi)$ are smooth, supported on the set Γ_j, and by Lemma 4.2.1 and the estimate (4.52) they satisfy the bounds

$$\left| F_{(j,\alpha)}(r, \xi) \right| \leq \left(\frac{C}{t} \right)^{(n-1)(4j+7)} \left(\frac{20}{r_{j+1} - r_j} \right)^{j+1} \tag{4.53}$$

Moreover we have precise knowledge of the lower order moments of $F_{(j,\alpha)}$ and bounds on those of higher order. Using the general spherical polar coordinates introduced in (4.45) we introduce the notation

$$M_{(j,\alpha),(k,\beta)} = \int_{\mathbb{R}^n} F_{(j,\alpha)}(r,\theta) r^k e^{i\beta\theta}\, d\sigma(\theta)\, dr$$

and can derive from (4.12) and (4.51) that

$$M_{(j,\alpha),(k,\beta)} = \begin{cases} 0 & \text{if some } |\beta_l| \leq 2j+2 \text{ and } \beta \neq \alpha \\ 0 & \text{if } k < j \\ 1 & \text{if } \beta = \alpha \text{ and } k = j \\ \mu_{j,k} & \text{if } \beta = \alpha \text{ and } k > j \end{cases} \tag{4.54}$$

where we have used the notation of (4.12) for the moments $\mu_{j,k}$ of $\psi_j(r)$. In the remaining case where all $|\beta_l| \geq 2j+3$ and $k \geq j$ we have from (4.52) that

$$\left|M_{(j,\alpha),(k,\beta)}\right| \leq \mu_{j,k} \left(\frac{C}{t}\right)^{(n-1)(4j+7)} \tag{4.55}$$

however in what follows we will only be interested in those moments $M_{(j,\alpha),(k,\beta)}$ for which $k \geq \max_l |\beta_l|$. For these moments it will be more useful to use $k \geq 2j+3$ to rewrite (4.55) as

$$\left|M_{(j,\alpha),(k,\beta)}\right| \leq \mu_{j,k} \left(\frac{C}{t}\right)^{4(n-1)(k-j-1)} \tag{4.56}$$

Construction

As in the one dimensional case (explained in Section 4.2) we proceed by inductively constructing a function with prescribed moments. Set $K^0(r,\theta) = F_{0,0}(r,\theta)$ and define (inductively)

$$N^j_{(k,\beta)} = \int_{\mathbb{R}^n} K^j(r,\theta) r^k e^{i\beta\theta}\, d\sigma(\theta)\, dr \tag{4.57}$$

$$K^{j+1}(r,\theta) = K^j(r,\theta) - \sum_{l=1}^{n-1} \sum_{|\alpha_l| \le j+1} N^j_{(j+1,\beta)} F_{(j+1,\alpha)}(r,\theta) \tag{4.58}$$

so that $N^{j+1}_{(j+1,\beta)} = 0$ for all β satisfying $|\beta_l| \le j+1$, $l = 1,\ldots,n-1$. By (4.54) the functions $F_{(j+1,\alpha)}$ do not affect the moments $N^{j+1}_{(k,\beta)}$ for $k \le j$, and consequently all of these lower order moments are zero.

$$N^{j+1}_{(k,\beta)} = \begin{cases} 1 & \text{if } k = 0 \text{ and } \beta = (0,\ldots,0) \\ 0 & \text{if } k \le j+1 \text{ and } |\beta_l| \le j+1 \text{ for } l = 1,\ldots,n-1 \end{cases} \tag{4.59}$$

There are finitely $F_{j,\alpha}$ for each j, all of which are supported on Γ_j. Since the sets Γ_j are disjoint it follows immediately that the above functions $K^j(x)$ have a pointwise limit function supported on Γ. In order for this to be of any interest we must have estimates that show the limit is integrable against polynomials and that its moments are given by the limit of the moments in (4.59).

Estimates

Our model is the estimation scheme for the one dimensional case that was described in Section 4.2. Notice that the moment sequence $N^j_{(k,\beta)}$ evolves according to the induction

$$N^{j+1}_{(k,\beta)} = N^j_{(k,\beta)} - \sum_{l=1}^{n-1} \sum_{|\alpha_l| \le j+1} N^j_{(j+1,\alpha)} M_{(j+1,\alpha),(k,\beta)} \tag{4.60}$$

As mentioned earlier, and implicit in our inductive definition (4.58), we are only interested in moments (k,β) for which $k \ge \max_l |\beta_l|$. In this situation we may compare (4.54) and

(4.56) to see that all of the moments $M_{(j+1,\alpha),(k,\beta)}$ occurring in the sum satisfy

$$\left|M_{(j+1,\alpha),(k,\beta)}\right| \leq \mu_{j+1,k}\left(\frac{C}{t}\right)^{4(n-1)(k-j-2)} \tag{4.61}$$

It is also easily seen that the number of terms in this sum is $(2j+3)^{n-1}$. These observations suggest defining a new sequence by

$$P_k^0 = \max\left\{\left|M_{(0,0),(k,\beta)}\right| : |\beta_l| \leq k \text{ for all } l = 1,\dots,n-1\right\} \tag{4.62}$$

$$P_k^{j+1} = P_k^j + P_{j+1}^j \mu_{j+1,k}\left(\frac{C_0}{t}\right)^{4(n-1)(k-j-2)} \tag{4.63}$$

where $C_0 = 2C$ is twice the constant in (4.61) and is fixed from here onward. The details of our previous work show that C_0 depends only upon the dimension n.

The benefit of this new sequence is that it dominates the sequence $N^j_{(k,\beta)}$ but will be much simpler to analyze. We record this as a lemma.

Lemma 4.4.1. *For all j, k, and β with $|\beta_l| \leq k$, $l = 0,\dots,n-1$ we have the bound*

$$\left|N^j_{(k,\beta)}\right| \leq P_k^j \tag{4.64}$$

Proof. For $j = 0$ this is obvious from the definition. Assuming the truth of the estimate for all superindices up to j we proceed inductively, looking at two cases. The simpler case is when $k \leq 2j+4$ whereupon $|\beta_l| \leq 2j+4$ and so by (4.54) all $M_{(j+1,\alpha),(k,\beta)} = 0$. Then

$$\begin{aligned}
\left|N^{j+1}_{(k,\beta)}\right| &= \left|N^j_{(k,\beta)} - \sum_{l=1}^{n-1}\sum_{|\alpha_l|\leq j+1} N^j_{(j+1,\alpha)} M_{(j+1,\alpha),(k,\beta)}\right| \\
&= \left|N^j_{(k,\beta)}\right| \\
&\leq P_k^j
\end{aligned}$$

$$\leq P_k^{j+1}$$

The more involved one has $k \geq 2j+5$. We use the bound (4.61) to obtain

$$\begin{aligned}
\left|N_{(k,\beta)}^{j+1}\right| &= \left|N_{(k,\beta)}^{j} - \sum_{l=1}^{n-1} \sum_{|\alpha_l| \leq j+1} N_{(j+1,\alpha)}^{j} M_{(j+1,\alpha),(k,\beta)}\right| \\
&\leq \left|N_{(k,\beta)}^{j}\right| + \left|\sum_{l=1}^{n-1} \sum_{|\alpha_l| \leq j+1} N_{(j+1,\alpha)}^{j}\right| \mu_{j+1,k} \left(\frac{C}{t}\right)^{4(n-1)(k-j-2)} \\
&\leq P_k^j + (2j+3)^{n-1} P_{j+1}^j \mu_{j+1,k} \left(\frac{C}{t}\right)^{4(n-1)(k-j-2)} \\
&\leq P_k^j + P_{j+1}^j \mu_{j+1,k} \left(\frac{C_0}{t}\right)^{4(n-1)(k-j-2)} \\
&= P_k^{j+1}
\end{aligned}$$

In the last step we used that $k \geq 2j+5$ whence $4(k-j-2) \geq 4j+12$ and so $(2j+3)^{n-1}$ is certainly dominated by $2^{(n-1)(4j+12)} = 2^{4(n-1)(k-j-2)}$. □

Our estimates for the sequence $\{P_k^j\}$ closely mimic those for the one dimensional case in Section 4.2. The key result is

Lemma 4.4.2. *The off-diagonal terms of the sequence $\{P_k^j\}$ satisfy the estimate*

$$P_{j+1}^j \leq C \frac{e^{2A(j-1)}}{A^{j-8}} \prod_{l=0}^{j} \mu_{l,l+1} \tag{4.65}$$

where $A = \left(\frac{C_0}{t}\right)^{4(n-1)}$ and C is independent of n and t.

Proof. Expanding P_k^{j+1} from the definition (4.63) we have

$$\begin{aligned}
P_k^{j+1} &= P_k^j + P_{j+1}^j \mu_{j+1,k} A^{(k-j-2)} \\
&= P_k^{j-1} + P_j^{j-1} \mu_{j,k} A^{(k-j-1)} + P_{j+1}^j \mu_{j+1,k} A^{(k-j-2)}
\end{aligned}$$

$$\vdots \tag{4.66}$$

$$= P_k^0 + P_1^0\mu_{1,k}A^{k-2} + P_2^1\mu_{2,k}A^{k-3} + \cdots + P_{j+1}^j\mu_{j+1,k}A^{(k-j-2)} \tag{4.67}$$

Recall the estimate (4.22) that stated

$$\frac{\mu_{j-1,k}}{\mu_{j-1,j}\mu_{j,k}} \leq \frac{2}{k-j+1}$$

and notice from (4.63) that

$$P_{l+1}^l = P_{l+1}^{l-1} + P_l^{l-1}\mu_{l,l+1}$$

whence

$$P_l^{l-1}\mu_{l,l+1} \leq P_{l+1}^l$$

Using these results we can compute part of the general term of (4.67)

$$\begin{aligned} P_l^{l-1}\mu_{l,k} &\leq \left(\frac{2}{k-l}\right)P_l^{l-1}\mu_{l,l+1}\mu_{l+1,k} \\ &\leq \left(\frac{2}{k-l}\right)P_{l+1}^l\mu_{l+1,k} \\ &\vdots \qquad \text{inductively} \\ &\leq \left(\frac{2}{k-l}\right)\left(\frac{2}{k-l-1}\right)\cdots\left(\frac{2}{k-j}\right)P_{j+1}^j\mu_{j+1,k} \\ &= \frac{(k-j-1)!2^{(j-l+1)}}{(k-l)!}P_{j+1}^j\mu_{j+1,k} \end{aligned} \tag{4.68}$$

It is also straightforward from (4.54), (4.55), and (4.22) to see that

$$\begin{aligned}P_k^0 &= \max\left\{\left|M_{(0,0),(k,\beta)}\right| : |\beta_l| \le k \text{ for all } l = 1,\dots,n-1\right\}\\ &\le A^7\mu_{0,k}\\ &\le A^7\left(\frac{2}{k}\right)\mu_{0,1}\mu_{1,k}\\ &\le A^7\left(\frac{2}{k}\right)P_1^0\mu_{1,k}\end{aligned}$$

so that applying (4.68) for the case $l = 1$ we have

$$P_k^0 \le A^7\frac{(k-j-1)!2^{(j+1)}}{k!}P_{j+1}^j\mu_{j+1,k} \tag{4.69}$$

Now we may substitute the estimates (4.68) and (4.69) into the expression (4.67) for P_k^{j+1} and obtain

$$\begin{aligned}P_k^{j+1} &= P_k^0 + \sum_{l=1}^{j+1} P_l^{l-1}\mu_{l,k}A^{(k-l-1)}\\ &\le \left[A^7\frac{(k-j-1)!2^{(j+1)}}{k!} + \sum_{l=1}^{j+1}\frac{(k-j-1)!2^{(j-l+1)}}{(k-l)!}A^{(k-l-1)}\right]P_{j+1}^j\mu_{j+1,k}\end{aligned}$$

We only need this result for the case $k = j+2$ where it reduces to

$$\begin{aligned}P_{j+2}^{j+1} &\le \left[\frac{A^7 2^{(j+1)}}{(j+2)!} + \sum_{l=1}^{j+1}\frac{(2A)^{(j-l+1)}}{(j+2-l)!}\right]P_{j+1}^j\mu_{j+1,j+2}\\ &= \left[\frac{A^7 2^{(j+1)}}{(j+2)!} + \frac{1}{2A}\sum_{m=1}^{j+1}\frac{(2A)^m}{m!}\right]P_{j+1}^j\mu_{j+1,j+2}\\ &\le \begin{cases}\frac{1}{2A}e^{2A}P_{j+1}^j\mu_{j+1,j+2} & \text{if } j \ge 6\\ \left(A^7 + \frac{1}{2A}e^{2A}\right)P_{j+1}^j\mu_{j+1,j+2} & \text{if } j < 6\end{cases}\end{aligned}$$

Providing $A \geq 10$ the above factor is bounded by (e^{2A}/A) independently of j, so inserting a small constant to resolve this case we can inductively reduce to

$$
\begin{aligned}
P^{j+1}_{j+2} &\leq C\frac{e^{2Aj}}{A^j} P^1_0 \prod_{l=1}^{j+1} \mu_{l,l+1} \\
&\leq C\frac{e^{2Aj}}{A^{j-7}} \prod_{l=0}^{j+1} \mu_{l,l+1}
\end{aligned}
$$

□

Properties of the Kernel

It was already mentioned that the inductive definition (4.58) involves only finitely many functions on each of the disjoint sets Γ_j and therefore has a pointwise limit function which we call $\tilde{K}(x)$. With the estimate (4.65) in hand we have a natural bound for $\tilde{K}(x)$ on Γ_{j+1}. From the definition (4.58) and the fact that all $F_{(l,\alpha)}(r,\xi)$ are zero on Γ_{j+1} except those with $l = j+1$, we see that

$$
\tilde{K}(x) = -\sum_{l=1}^{n-1} \sum_{|\alpha_l| \leq j+1} N^j_{(j+1,\beta)} F_{(j+1,\alpha)}(r,\xi)
$$

on the set Γ_{j+1}. Using (4.64) this gives

$$
\left|\tilde{K}(x)\right| \leq (2j+3)^{n-1} P^j_{j+1} \left|F_{(j+1,\alpha)}(r,\xi)\right|
$$

so that substituting the bounds (4.53) and (4.65) (writing both in terms of A), then using (4.14) gives

$$
\left|\tilde{K}(x)\right| \leq C(2j+3)^{n-1} \frac{e^{2A(j-1)}}{A^{j-8}} \left(\frac{A}{2^{4(n-1)}}\right)^{j+1} \left(\frac{20}{r_{j+2}-r_{j+1}}\right)^{j+2} \prod_{l=0}^{j} \mu_{l,l+1}
$$

$$\leq \frac{C}{A^7} e^{2A(j-1)} \left(\frac{20}{r_{j+2} - r_{j+1}} \right)^{j+2} \prod_{l=0}^{j} (l+1) r_l$$
$$= \frac{C}{A^7} e^{2A(j-1)} (j+1)! \left(\prod_{l=0}^{j} r_l \right) \left(\frac{20}{r_{j+2} - r_{j+1}} \right)^{j+2}$$

This is now very similar to the situation encountered in our one dimensional construction (see Section 4.2, particularly (4.27)). If we set

$$r_j = T \exp\left[2 \log^2 (j + j_0)\right] \tag{4.70}$$

then we can directly apply the estimate (4.29) of Lemma 4.2.4 to obtain on the set Γ_j

$$\begin{aligned} \log\left|\tilde{K}(x)\right| &\leq C - 7 \log A + 2A(j-2) + 2 j_0 \log^2(j + j_0) - 2(j + j_0) \log(j + j_0) \\ &\leq -(j + j_0 + 1) \log(j + j_0 + 1) \end{aligned} \tag{4.71}$$

for all sufficiently large j. By the definition (4.70) we also know that $\log |x| \leq \log T + 2 \log^2(j + j_0 + 1)$ on Γ_j, so that

$$\log(j + j_0 + 1) \geq \left(\frac{1}{2} \log \frac{|x|}{T} \right)^{1/2}$$

and therefore

$$\log\left|\tilde{K}(x)\right| \leq - \left(\frac{1}{2} \log \frac{|x|}{T} \right)^{1/2} \exp\left(\frac{1}{2} \log \frac{|x|}{T} \right)^{1/2} \tag{4.72}$$

for all sufficiently large $|x|$. This rate of decay ensures $\tilde{K}(x)$ is integrable against all functions having at most polynomial growth in the variable $|x|$, and by the construction (see

(4.59)) and the dominated convergence theorem we have

$$\int_{\mathbb{R}^n} \tilde{K}(r,\xi) r^k e^{i\beta\theta}\, d\sigma(\theta)\, dr = \begin{cases} 1 & \text{if } k = 0 \text{ and } \beta = (0,\dots,0) \\ 0 & \text{if } k \in \mathbb{N} \setminus \{0\} \text{ and all } |\beta_l| \le k \end{cases} \tag{4.73}$$

At this point we pause to recognize that (4.73) implies the function $\tilde{K}(x)$ has zero polynomial moments except for the moment corresponding to the constant function. This is because all polynomials in $x_1, \dots, x_n$ may be expressed in terms of functions $r^k e^{i\beta\theta}$.

Lemma 4.4.3. *Any monomial x^α may be written*

$$x^\alpha = r^{|\alpha|} \sum_\beta a_\beta e^{i\beta\theta} \tag{4.74}$$

where $r = |x|$ and each β occurring in the sum satisfies $|\beta_l| \le |\alpha|$ for $l = 1, 2, \dots, n$.

Proof. We write

$$\frac{x^\alpha}{r^{|\alpha|}} = \frac{x^\alpha}{|x|^{|\alpha|}} = \xi^\alpha \tag{4.75}$$

where ξ is a point on S^{n-1}. Recall from (4.45) that

$$\xi_j = \begin{cases} \cos\theta_1 & \text{if } j = 1 \\ \cos\theta_j \prod_{l=1}^{j-1} \sin\theta_l & \text{if } 1 < j < n \\ \prod_{l=1}^{n-1} \sin\theta_l & \text{if } j = n \end{cases}$$

so that

$$\xi^\alpha = (\cos^{\alpha_1}\theta_1) \prod_{j=1}^{n-1} \left(\cos\theta_j \prod_{l=1}^{j-1} \sin\theta_l \right)^{\alpha_j} \left(\prod_{l=1}^{n-1} \sin\theta_l \right)^{\alpha_n} \tag{4.76}$$

which is a polynomial in $e^{i\beta\theta}$ after substituting

$$\cos\theta_j = \frac{e^{i\theta_j} + e^{i\theta_j}}{2} \qquad\qquad \sin\theta_j = \frac{e^{i\theta_j} - e^{i\theta_j}}{2i}$$

and in conjunction with (4.75) gives a representation of the form of (4.74). We note in particular that the variable θ_j occurs in (4.76) only as

$$\begin{aligned}(\cos\theta_j)^{\alpha_j}\prod_{l=j+1}^{n-1}(\sin\theta_j)^{\alpha_l} &= (\cos\theta_j)^{\alpha_j}(\sin\theta_j)^{\alpha_{j+1}+\cdots+\alpha_{n-1}}\\ &= \left(\frac{e^{i\theta_j} + e^{i\theta_j}}{2}\right)^{\alpha_j}\left(\frac{e^{i\theta_j} - e^{i\theta_j}}{2i}\right)^{\alpha_{j+1}+\cdots+\alpha_{n-1}}\end{aligned}$$

so that for each β in (4.74) we have

$$|\beta_j| \leq \sum_{l\geq j}|\alpha_l| \leq |\alpha|$$

$\square$

As a consequence of Lemma 4.4.3 we conclude from (4.73) that

$$\int_{\mathbb{R}^n}\tilde{K}(x)x^\alpha\, d\sigma(\theta)\, dr = \begin{cases}1 & \text{if } \alpha = (0,\ldots,0)\\ 0 & \text{if } \alpha \in \mathbb{N}^n \setminus \{(0,\ldots,0)\}\end{cases}$$

Since x^α is a real-valued function the same is true when $\tilde{K}(x)$ is replaced by its real part $\mathrm{Re}(\tilde{K})$. Adjusting by the factor $|x|^{n-1}$ that relates $d\sigma(\theta)\, dr$ to dx we define

$$K(x) = \frac{\mathrm{Re}(\tilde{K}(x))}{|x|^{n-1}}$$

which is a smooth function supported on $\Gamma = \mathrm{Sppt}(\tilde{K})$, satisfies

$$\int_{\mathbb{R}^n} K(x)x^\alpha \, dx = \begin{cases} 1 & \text{if } \alpha = (0, \ldots, 0) \\ 0 & \text{if } \alpha \in \mathbb{N}^n \setminus \{(0, \ldots, 0)\} \end{cases}$$

and is bounded by $\frac{|\tilde{K}(x)|}{|x|^{n-1}}$. The estimate (4.72) gives a bound for the decay of $K(x)$ when $|x|$ is large, so there is a constant $C = C(n, t, j_0, T)$ such that

$$\left|K(x)\right| \leq \frac{C}{|x|^{n-1}} \exp\left[-\left(\frac{1}{2}\log\frac{|x|}{T}\right)^{1/2} \exp\left(\frac{1}{2}\log\frac{|x|}{T}\right)^{1/2}\right]$$

This completes the proof of Lemma 4.0.4 and therefore Theorem 4.0.3.

Chapter 5

Proof of the Main Theorem

The goal of this chapter is to prove Theorem 2.1.1. Following the method outlined in Section 2.2 we define an extension operator as a smooth sum of operators corresponding to cubes. The operator for a cube Q is constructed in Section 5.2 and involves convolution against a polynomial reproducing kernel of the type introduced in Chapter 4 and supported on one of the twisting cones discussed in Chapter 3. Section 5.3 then deals with proving that this operator takes $f \in W^{k,p}(\Omega)$ to $\mathcal{E}f \in W^{k,p}((\Omega^c)^o)$ by establishing the estimates described in Section 2.2, and in Section 5.4 we show that the result is an extension of f.

5.1 An Elementary Reduction

From the results of Chapter 3 we have a good understanding of the geometry of that part of Ω which lies close to $\partial\Omega$. All of our constructions will involve this geometry and therefore only be applicable in this region. However this is not really a restriction on our method because the problem of extending $f \in W^{k,p}(\Omega)$ is in a natural sense a local problem near $\partial\Omega$. The simplest way to see this is from the following lemma

Lemma 5.1.1. *Given $\lambda > 0$ and $f \in W^{k,p}(\Omega)$ there is $g \in W^{k,p}(\Omega)$ such that*

- *The support of g is in* $\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \leq \lambda\}$,
- *The functions f and g are equal on the set* $\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \leq \lambda/2\}$
- *There is an estimate*

$$||g||_{W^{k,p}(\Omega)} \leq C(\lambda, k)||f||_{W^{k,p}(\Omega)}$$

Proof. Let $\chi(x)$ be a C^∞ function supported on $B(0, 1)$ and with $\int \chi = 1$. Then $\chi_t = t^{-n}(x/t)$ is C^∞, supported on $B(0, t)$ and satisfies $|\nabla^m \chi| \leq C(m)t^{-m}$. Convolution of the characteristic function of $\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \leq 3\lambda/4\}$ with $\chi_{\lambda/4}(x)$ then gives a function $\Phi \in C^\infty$ such that

$$\Phi(y) \equiv \begin{cases} 1 & \text{on } \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \leq \lambda/2\} \\ 0 & \text{on } \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geq \lambda\} \end{cases}$$

and with the estimates $|\nabla^m \Phi| \leq C(m)\lambda^{-m}$ on the remaining piece of Ω. The product $g(x) = f(x)\Phi(x)$ now has derivatives

$$D^\alpha g(x) = \sum_{0 \leq \beta \leq \alpha} D^\beta f(x) D^{\alpha-\beta}\Phi(x)$$

and therefore

$$\begin{aligned} ||D^\alpha g(x)||_{L^p} &\leq \sum_{0 \leq \beta \leq \alpha} ||D^{\alpha-\beta} f||_{L^p} C(|\alpha - \beta|)\lambda^{-|\alpha-\beta|} \\ &\leq C(\lambda, k)||f||_{W^{k,p}(\Omega)} \end{aligned}$$

for $|\alpha| \leq k$, as was required. □

An immediate consequence of Lemma 5.1.1 is that if we can define a function $\mathcal{E}g$ on

Ω^c such that

$$G(x) = \begin{cases} g(x) & \text{for } x \in \Omega \\ \mathcal{E}g(x) & \text{for } x \in \Omega^c \end{cases}$$

is in $W^{k,p}(\mathbb{R}^n)$ with $||G||_{W^{k,p}(\mathbb{R}^n)} \leq C||g||_{W^{k,p}(\Omega)}$ then

$$F(x) = \begin{cases} f(x) & \text{for } x \in \Omega \\ \mathcal{E}g(x) & \text{for } x \in \Omega^c \end{cases}$$

also has $||F||_{W^{k,p}(\mathbb{R}^n)} \leq C(\lambda, k)||f||_{W^{k,p}(\Omega)}$, so the problem of extension for $W^{k,p}(\Omega)$ need only involve functions supported on a small neighborhood of $\partial\Omega$.

5.2 The Extension Operator

We wish to define our extension operator as a smooth sum of operators $\mathcal{E}_Q$, where each $\mathcal{E}_Q$ is convolution with a polynomial reproducing kernel supported on a twisting cone corresponding to Q. The fact that such a kernel necessarily has unbounded support inevitably introduces some technicalities. They are mostly dealt with by smoothly cutting off f at some distance from $\partial\Omega$ as in Section 5.1, however we also need a preliminary construction of an unbounded twisting cone corresponding to a cube.

The Cone and Kernel for a Small Cube

Let $\mathcal{W}_1$ be the Whitney cubes from $(\Omega^c)^o$ such that $l(Q) \leq \epsilon\delta/200n$ and fix $Q \in \mathcal{W}_1$. Corresponding to this cube we may take a chain $\{S_j\}$ of Whitney cubes of Ω with properties as in Lemma 3.2.3. Within the chain $\{S_j\}$ we have a twisting cone Γ_Q as constructed in Chapter 3 Section 3.3.

In order to apply the results of Chapter 4 we translate Γ_Q to the origin and rescale by $l(Q)^{-1}$. This sort of translating and rescaling will occur several times during the proof so we take this opportunity to fix some notation. Unadorned variables and sets x, y, Γ_Q will be in the usual space $\mathbb{R}^n$, while symbols decorated with a tilde, like $\tilde{x}$, $\tilde{y}$, and $\tilde{\Gamma}_Q$, will refer to the corresponding objects in the (dimensionless) parameter space $\mathbb{R}^n$. The relevant transformation is $\tilde{x} = (x - x_Q)/l(Q)$, and our first use of it is

$$\tilde{\Gamma}_Q = \frac{1}{l(Q)}(\Gamma_Q - x_Q)$$

Recall from Lemma 3.2.3 that the radius of the twisting cone Γ_Q grows linearly with the distance from Q for some range of scales. Rescaling this to $\tilde{\Gamma}_Q$ we see that there is an inner radius R_0, an outer radius $R_1(l(Q))^{-1}$, and a constant t such that for all $\tilde{r} \in [R_0, R_1(l(Q))^{-1}]$ there is $\tilde{y}$ with $|\tilde{y}| = \tilde{r}$ and

$$B(\tilde{y}, t|\tilde{y}|) \subset \tilde{\Gamma}_Q \tag{5.1}$$

Each of the constants R_0, R_1 and t depends only on n, ϵ, and δ. In particular we note for later reference that in Lemma 3.2.3 we had a cube of size $\epsilon\delta/10\sqrt{n}$ at radius R_1 from x_Q and so by (1.1) we can take $R_1 = \epsilon\delta/10$.

If (5.1) were true also for $\tilde{r} \geq R_1(l(Q))^{-1}$ then Theorem 4.0.3 could be applied to produce a reproducing kernel for polynomials on $\tilde{\Gamma}_Q$. To make this possible we will adjoin a piece of cone to $\tilde{\Gamma}_Q$ in the following manner.

Apply (5.1) to find $\tilde{y}$ with $|\tilde{y}| = R_1(l(Q))^{-1}$ and $B = B(\tilde{y}, t|\tilde{y}|) \subset \tilde{\Gamma}_Q$. The set we attach to $\tilde{\Gamma}_Q$ is the unbounded piece of cone over $B \cap R_1(l(Q))^{-1}S^{n-1}$ with vertex at the origin. This may be written

$$\left\{\tilde{x} : |\tilde{x}| \geq R_1(l(Q))^{-1} \text{ and } \frac{R_1(l(Q))^{-1}}{|x|}x \in B(\tilde{y}, t|\tilde{y}|)\right\}$$

In order to avoid some technical issues later we trim extraneous material from $\tilde{\Gamma}_Q$ as well as attaching the new piece. With $\tilde{y}$ as above define

$$\tilde{\Gamma}^*_Q = \left(\tilde{\Gamma}_Q \cap \{R_0 \le |\tilde{x}| \le R_1(l(Q))^{-1}\}\right) \cup \left\{\tilde{x} : |\tilde{x}| \ge R_1(l(Q))^{-1} \text{ and } \frac{R_1(l(Q))^{-1}}{|x|} x \in B(\tilde{y}, t|\tilde{y}|)\right\}$$

In keeping with our notation we also define

$$\Gamma^*_Q = l(Q)(\tilde{\Gamma}^*_Q + x_Q)$$

The result of the construction so far is illustrated in Figure 5.2.

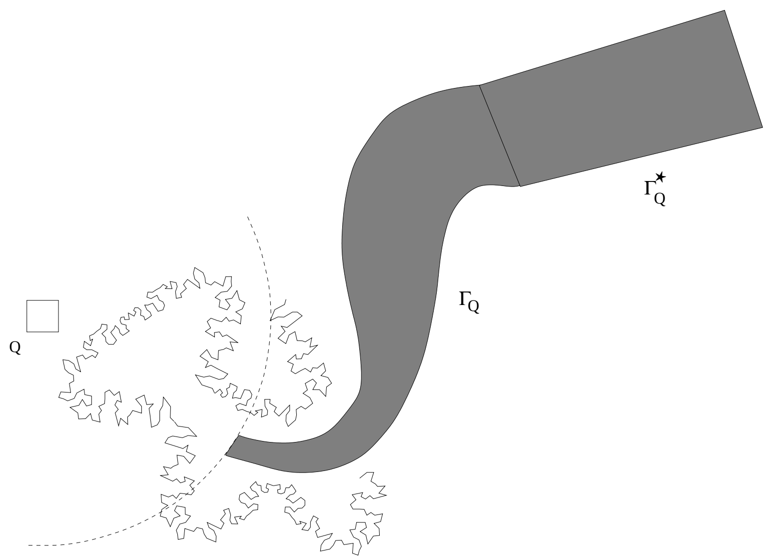

Figure 5.1: The set Γ^*_Q

We record for future reference a trivial consequence of Lemma 3.3.1.

Lemma 5.2.1. *If $\tilde{y} \in \tilde{\Gamma}_Q$ and $x \in (17/16)Q$ then $(x + l(Q)\tilde{y}) \in S_j$ for some S_j in the chain covering Γ_Q. In fact if $\tilde{y}$ is such that $(x_Q + l(Q)\tilde{y}) \in \Gamma_Q \cap S_j$ then $(x + l(Q)\tilde{y}) \in S_{j-1} \cup S_j \cup S_{j+1}$.*

Now $\tilde{\Gamma}^*_Q$ has the property that for all $\tilde{r} \ge R_0$ there is $\tilde{y}$ with $|\tilde{y}| = \tilde{r}$ and

$$B(\tilde{y}, t|\tilde{y}|) \subset \tilde{\Gamma}^*_Q$$

Applying Theorem 4.0.3 we then have a smooth function $\tilde{K}_Q(\tilde{y})$ supported on $\tilde{\Gamma}^*_Q$ and having the properties

$$\int_{\mathbb{R}^n} \tilde{y}^\alpha \tilde{K}_Q(\tilde{y}) = \begin{cases} 1 & \text{if } \alpha = (0,\dots,0) \\ 0 & \text{if } \alpha \in \mathbb{N}^n \setminus \{(0,\dots,0)\} \end{cases} \tag{5.2}$$

$$\left|\tilde{K}_Q(\tilde{y})\right| \leq \frac{C}{|\tilde{y}|^{n-1}} \exp\left[-\left(\frac{1}{2}\log\frac{|\tilde{y}|}{T}\right)^{1/2} \exp\left(\frac{1}{2}\log\frac{|\tilde{y}|}{T}\right)^{1/2}\right] \tag{5.3}$$

where C and T are constants depending only on R_0, R_1, and t, and therefore only on n, ϵ, and δ. It will be convenient later to have simpler notation for (5.3) and to know a variant of it on cubes S_j of the chain containing Γ_Q. We therefore record that if $x \in (17/16)Q$ and $y \in S_j$ then by Lemma 5.2.1 and the linear growth (3.5) of the chain $\{S_j\}$

$$\left|\tilde{K}_Q\left(\frac{y-x}{l(Q)}\right)\right| \leq \left(\frac{l(Q)}{l(S_j)}\right)^{n-1} \kappa\left(\frac{l(S_j)}{l(Q)}\right) \tag{5.4}$$

where

$$\kappa(s) = C\exp\left[-\left(\frac{1}{2}\log\frac{s}{T}\right)^{1/2} \exp\left(\frac{1}{2}\log\frac{s}{T}\right)^{1/2}\right] \tag{5.5}$$

Definition of the Operator

Let $f \in L^1_{\text{loc}}(\Omega)$. To accommodate the restriction that we must work on a small neighborhood of $\partial\Omega$, we first multiply f by the C^∞ cutoff function introduced in Section 5.1 with $\lambda = \epsilon\delta/100n$. Somewhat abusing notation we also use f to denote the resulting function, which now vanishes identically on any sufficiently large Whitney cube S.

$$f \equiv 0 \text{ on } S \text{ if } l(S) \geq \frac{\epsilon\delta}{100\sqrt{n}} \tag{5.6}$$

Fix $Q \in \mathcal{W}_1$. In essence we wish to define $\mathcal{E}_Q f$ on $(17/16)Q$ by convolution of f against $\tilde{K}_Q$ with a scaling parameter of $l(Q)$, however a slight difficulty is introduced by the fact that f may be undefined at points of $\Gamma_Q^* \setminus \Gamma_Q$. To avoid this annoyance we cut off f outside the set of interest, which in this case is all x such that $|\tilde{x}| \leq R_1(l(Q))^{-1}$. Let

$$f_Q(x) = \begin{cases} f(x) & \text{if } |x - x_Q| \leq R_1 \\ 0 & \text{otherwise} \end{cases} \tag{5.7}$$

It is worth noting that the use of a characteristic function to cut f here will not be a problem because it occurs at the fixed radius R_1 from x_Q. At this radius the cubes S_j covering Γ_Q have length at least $\epsilon\delta/(10\sqrt{n})$ and we already know from (5.6) that $f \equiv 0$ on these cubes. The function f_Q is therefore a C^∞ continuation of f from Γ_Q to Γ_Q^*.

Define the operator $\mathcal{E}_Q f(x)$ for $x \in (17/16)Q$ by

$$\mathcal{E}_Q f(x) = \begin{cases} \displaystyle\int_{\mathbb{R}^n} f_Q(x + l(Q)\tilde{y})\tilde{K}_Q(\tilde{y})\, d\tilde{y} & \text{if } Q \in \mathcal{W}_1 \\ 0 & \text{otherwise} \end{cases} \tag{5.8}$$

Note from the preceding discussion that the convolution really only involves f on a small neighborhood of Γ_Q. In particular it follows from Lemma 5.2.1 and the fact $f_Q \equiv 0$ on $\Gamma_Q^* \setminus \Gamma_Q$ that the convolution in (5.8) only involves values of f_Q on $\cup S_j$, where in particular f_Q coincides with f. (To see this last statement is true on the largest cubes from the chain $\{S_j\}$ we must again use that $f \equiv 0$ on these cubes by (5.6).)

Finally we define the extension operator at all points in $(\Omega^c)^o$ by

$$\mathcal{E}_Q f(x) = \sum_Q \mathcal{E}_Q f(x)\Phi_Q(x) \tag{5.9}$$

as previewed in (2.4) of Section 2.2. The functions $\Phi_Q(x)$ are the smooth partition of unity introduced in Lemma 2.2.1. We also define $\mathcal{E}f(x) = f(x)$ at all points $x \in \Omega$. By Lemma 1.1.4 the boundary $\partial\Omega$ has no measure, so $\mathcal{E}f$ is defined almost everywhere.

5.3 Estimates for the Extension Operator

The purpose of this section is to prove that our operator gives a function in the correct space on $(\Omega^c)^o$. We state this as a theorem.

Theorem 5.3.1. *For fixed $k \in \mathbb{N}$, $1 \leq p \leq \infty$, and $f \in W^{k,p}(\Omega)$, the function $\mathcal{E}f$ is in $W^{k,p}((\Omega^c)^o)$ with the estimate*

$$\left\|\mathcal{E}f\right\|_{W^{k,p}\left((\Omega^c)^o\right)} \leq C(n,\epsilon,\delta,k,p)\left\|f\right\|_{W^{k,p}(\Omega)} \tag{5.10}$$

Proof. The first step in defining $\mathcal{E}f$ was to replace f by the product of f with the smooth cutoff function introduced in Section 5.1. We see from Lemma 5.1.1 that the $W^{k,p}(\Omega)$ norm of the product is comparable to that of f and therefore it suffices to prove the bound (5.10) for this new function. By the discussion following that lemma it is also clear that any extension of the product is also an extension of f, so we may henceforth ignore this step in the definition and simply assume that $f \equiv 0$ on cubes of length at least $\epsilon\delta/(100\sqrt{n})$.

Suppose that $1 \leq p < \infty$. By the argument given in Section 2.2 we have the bound

$$\begin{aligned}
&||D^\alpha \mathcal{E}f||^p_{L^p\left((\Omega^c)^o\right)} && (5.11)\\
&= \sum_{Q' \in \mathcal{W}} ||D^\alpha \mathcal{E}f||^p_{L^p(Q')}\\
&\leq C_1 \sum_{Q' \in \mathcal{W}} ||D^\alpha \mathcal{E}_{Q'} f||^p_{L^p(Q')}\\
&\quad + C_2 \sum_{Q' \in \mathcal{W}} \sum_{Q \in \mathcal{N}(Q')} \sum_{0 \leq \beta \leq \alpha} c(|\alpha-\beta|)^p l(Q')^{-|\alpha-\beta|p} ||D^\beta(\mathcal{E}_Q f - \mathcal{E}_{Q'} f)||^p_{L^p(Q' \cap (17/16)Q)} && (5.12)
\end{aligned}$$

for the operator defined by (5.9). The constants C_1 and C_2 depend only on n, k and p. Inserting the estimates proved in Lemma 5.3.4 and Lemma 5.3.5 below we obtain

$$\begin{aligned}\left\|D^\alpha \mathcal{E} f\right\|^p_{L^p((\Omega^c)^o)} &\leq C\left\|D^\alpha f(z)\right\|^p_{L^p(\Omega)} + C\left\|\nabla^k f(y)\right\|^p_{L^p(\Omega)} \\ &\leq C\|f\|^p_{W^{k,p}(\Omega)}\end{aligned}$$

where our constants now depend also on ϵ and δ. This completes the proof for $1 \leq p < \infty$ because

$$\begin{aligned}\left\|\mathcal{E} f\right\|_{W^{k,p}((\Omega^c)^o)} &= \sum_{|\alpha|\leq k}\left\|D^\alpha \mathcal{E} f\right\|_{L^p((\Omega^c)^o)} \\ &\leq C(n,\epsilon,\delta,k,p)\|f\|_{W^{k,p}(\Omega)}\end{aligned}$$

The proof for $p = \infty$ is also based on the estimates in Lemma 5.3.4 and Lemma 5.3.5, but in this case we use (2.6) of Chapter 2, Section 2.2. This gives the pointwise bound at $x \in Q'$

$$\begin{aligned}|D^\alpha \mathcal{E} f(x)| &\leq |D^\alpha \mathcal{E}_{Q'} f(x)| + \sum_{Q\in\mathcal{N}(Q')}\sum_{0\leq\beta\leq\alpha} c(|\alpha-\beta|)l(Q')^{-|\alpha-\beta|}|D^\beta(\mathcal{E}_Q f(x) - \mathcal{E}_{Q'} f(x))| \\ &\leq C\|D^\alpha f\|_{L^\infty(\Omega)} + C\|\nabla^k f\|_{L^\infty(\Omega)} l(Q')^{k-|\alpha|} \\ &\leq C\|f\|_{W^{k,p}(\Omega)}\end{aligned}$$

where we have used that Q' has finitely many neighbors and that the length of cubes $Q' \in \mathcal{W}_1$ are bounded. □

Useful Estimates for $\tilde{K}_Q$

As we proceed with the proof we will have need of some estimates for sums and integrals of $\tilde{K}_Q$. To assist in the flow of the material and avoid repetition we list them here.

Lemma 5.3.2. *With $\kappa(t)$ as defined in (5.5) we have*

$$\sum_{j=m}^{\infty} 2^{qj}\kappa(2^j) \le C_1 2^{qm}\kappa(2^m)$$
$$\sum_{j=0}^{\infty} 2^{qj}\kappa(2^j) \le C_2$$

where C_1 and C_2 depend on n, ϵ, δ, and q but not on m.

Proof. Using the definition (5.5) of $\kappa(t)$ we see that there are constants c_1, c_2, and c_3 depending only on n, ϵ and δ, such that we may bound the sum by an integral

$$\begin{aligned}
\sum_{j=m}^{\infty} 2^{qj}\kappa(2^j) &\le c_1 \int_m^{\infty} \exp\left[c_2 qt - c_3 t^{1/2} e^{c_3 t^{1/2}}\right] dt \\
&= 2^{qm}\kappa(2^m) c_1 \int_m^{\infty} \exp\left[c_2 q(t-m) - c_3\left(t^{1/2}e^{c_3 t^{1/2}} - m^{1/2}e^{c_3 m^{1/2}}\right)\right] dt \\
&= 2^{qm}\kappa(2^m) c_1 \int_0^{\infty} \exp\left[c_2 qs - c_3\left((s+m)^{1/2}e^{c_3 (s+m)^{1/2}} - m^{1/2}e^{c_3 m^{1/2}}\right)\right] ds
\end{aligned}$$

It is clear this integral is finite for any $m \ge 0$ and q, with a bound $C(m,q)$ depending continuously on m. However if $m > c_3^{-2}$ then convexity implies

$$c_3(s+m)^{1/2}e^{c_3(s+m)^{1/2}} - c_3 m^{1/2}e^{c_3 m^{1/2}} \ge c_3 s e^{c_3 s^{1/2}} - e$$

so that in this case the integral term is bounded by

$$\int_0^{\infty} \exp\left[c_2 qs - c_3 s^{1/2} e^{c_3 s^{1/2}} + e\right] ds \le C(q)$$

and we conclude that the integral is always bounded by the larger of $C(q)$ and the maximum of $C(m, q)$ over $m \in [0, c_3^{-2}]$. □

Corollary 5.3.3.

$$\int_{\mathbb{R}^n} \left|\tilde{K}_Q(\tilde{y})\right| d\tilde{y} \leq C(n, \epsilon, \delta)$$

Proof. Simply integrate radially by dividing $\mathbb{R}^n$ up into concentric annuli from radius 2^j to 2^{j+1}. From (5.3) and (5.5) we see immediately that

$$\int_{\mathbb{R}^n} \left|\tilde{K}_Q(\tilde{y})\right| d\tilde{y} \leq C \sum_{j=0}^{\infty} 2^j \kappa(2^j)$$

and the result follows from Lemma 5.3.2. □

Estimates for Individual Cubes

The simpler of the estimates we need concerns the behavior of the operator $\mathcal{E}_Q$ on the cube Q. We state it as a lemma

Lemma 5.3.4. *If $\mathcal{E}_Q$ is the operator defined in* (5.8) *then for* $1 \leq p < \infty$

$$\sum_{Q \in \mathcal{W}} \|D^\alpha \mathcal{E}_Q f\|^p_{L^p(Q)} \leq C \left\|D^\alpha f(z)\right\|^p_{L^p(\Omega)} \tag{5.13}$$

and when $p = \infty$

$$\|D^\alpha \mathcal{E} f\|_{L^\infty(Q)} \leq C \|D^\alpha f\|_{L^\infty(\Omega)}$$

where $C = C(n, \epsilon, \delta, k, p)$.

Proof. The estimate is trivial for those cubes where $\mathcal{E}_Q$ is identically zero, so we may restrict our attention to the cubes where it is given by the integral in (5.8). As f and its derivatives are locally integrable and $\tilde{K}_Q$ has rapid decay we may differentiate within the integral to obtain

$$D^\alpha \mathcal{E}_Q f(x) = \int_{\mathbb{R}^n} D^\alpha f_Q(x + l(Q)\tilde{y})\tilde{K}_Q(\tilde{y})\, d\tilde{y} \tag{5.14}$$

It could be objected that f_Q might have very bad derivatives on the circle $|x - x_Q| = R_1$ where we cut it off by a characteristic function, however this is not an issue for the same reason given in the comments following (5.7), specifically the fact that $f \equiv 0$ in a neighborhood of $\Gamma_Q^* \cap \{|x - x_Q| = R_1\}$.

We can now quickly deal with the case $p = \infty$. The discussion following (5.8) showed that the only points $(x + l(Q)\tilde{y})$ where the integrand is non-zero are in the chain $\{S_j\}$ of Whitney cubes containing the twisting cone Γ_Q, where in particular $f_Q \equiv f$. Therefore if $f \in W^{k,\infty}(\Omega)$ we can apply Corollary 5.3.3 to obtain

$$\begin{aligned} \left|D^\alpha \mathcal{E}_Q f(x)\right| &= \left|\int_{\mathbb{R}^n} D^\alpha f_Q(x + l(Q)\tilde{y})\tilde{K}_Q(\tilde{y})\, d\tilde{y}\right| \\ &\leq \left\|D^\alpha f\right\|_{L^\infty(\Omega)} \int_{\mathbb{R}^n} \left|\tilde{K}_Q(\tilde{y})\right| d\tilde{y} \\ &\leq C \left\|D^\alpha f\right\|_{L^\infty(\Omega)} \end{aligned}$$

with a constant $C = C(n, \epsilon, \delta)$. For the remainder of the proof we will therefore assume that $1 \leq p < \infty$.

Hölder's inequality and Corollary 5.3.3 may be applied to (5.14) to yield

$$\begin{aligned} \left|D^\alpha \mathcal{E}_Q f(x)\right| &\leq \left(\int_{\mathbb{R}^n} \left|D^\alpha f_Q(x + l(Q)\tilde{y})\right|^p \left|\tilde{K}_Q(\tilde{y})\right| d\tilde{y}\right)^{1/p} \left(\int_{\mathbb{R}^n} \left|\tilde{K}_Q(\tilde{y})\right| d\tilde{y}\right)^{(p-1)/p} \\ &\leq C \left(\int_{\mathbb{R}^n} \left|D^\alpha f_Q(x + l(Q)\tilde{y})\right|^p \left|\tilde{K}_Q(\tilde{y})\right| d\tilde{y}\right)^{1/p} \end{aligned}$$

so we have

$$\left\|D^{\alpha}\mathcal{E}_Q f\right\|^p_{L^p(Q)} \leq C \int_Q \int_{\mathbb{R}^n} \left|D^{\alpha} f_Q(x + l(Q)\tilde{y})\right|^p \left|\tilde{K}_Q(\tilde{y})\right| d\tilde{y}\, dx \tag{5.15}$$

Now if we make a change of variables

$$\int_{\mathbb{R}^n} \left|D^{\alpha} f_Q(x + l(Q)\tilde{y})\right|^p \left|\tilde{K}_Q(\tilde{y})\right| d\tilde{y} = \frac{1}{l(Q)^n} \int_{\mathbb{R}^n} \left|D^{\alpha} f_Q(z)\right|^p \left|\tilde{K}_Q\left(\frac{z-x}{l(Q)}\right)\right| dz$$

then using Lemma 5.2.1 and the fact $f_Q \equiv 0$ on Γ^*_Q we see that the support of the integrand is contained in $\cup S_j$. Applying (5.4) to estimate $\left|\tilde{K}_Q((z-x)/l(Q))\right|$ for points $z \in S_j$ and $x \in Q$ we may then write

$$\begin{aligned}
\left\|D^{\alpha}\mathcal{E}_Q f\right\|^p_{L^p(Q)} &\leq C \frac{1}{l(Q)^n} \int_Q \sum_j \left(\frac{l(Q)}{l(S_j)}\right)^{n-1} \kappa\left(\frac{l(S_j)}{l(Q)}\right) \int_{S_j} \left|D^{\alpha} f_Q(z)\right|^p dz\, dx \\
&\leq C \sum_j \left(\frac{l(Q)}{l(S_j)}\right)^{n-1} \kappa\left(\frac{l(S_j)}{l(Q)}\right) \int_{S_j} \left|D^{\alpha} f(z)\right|^p dz
\end{aligned}$$

because the integrand is then independent of $x \in Q$, and $f_Q \equiv f$ on $\cup S_j$.

It is now possible to sum over all $Q \in \mathcal{W}_1$ as is needed for (5.13). We use the notation introduced in Section 3.3. Let $\mathcal{G}(S)$ be the set of all cubes $Q \in \mathcal{W}_1$ such that the twisting cone corresponding to Q intersects the Whitney cube S of Ω. and recall (3.10) in which we bounded the number of cubes of size $l(Q) = 2^{-m} l(S)$ in $\mathcal{G}(S)$ by $C(\epsilon)2^{nm}$. This yields

$$\begin{aligned}
\sum_{Q\in\mathcal{W}_1} \left\|D^{\alpha}\mathcal{E}_Q f\right\|^p_{L^p(Q)} &\leq C \sum_{Q\in\mathcal{W}_1} \sum_{S_j\cap\Gamma_Q} \left(\frac{l(Q)}{l(S_j)}\right)^{n-1} \kappa\left(\frac{l(S_j)}{l(Q)}\right) \int_{S_j} \left|D^{\alpha} f(z)\right|^p dz \\
&\leq C \sum_{S\in\mathcal{W}(\Omega)} \left\|D^{\alpha} f(z)\right\|^p_{L^p(S)} \sum_{Q\in\mathcal{G}(S)} \left(\frac{l(Q)}{l(S)}\right)^{n-1} \kappa\left(\frac{l(S)}{l(Q)}\right) \\
&\leq C \sum_{S\in\mathcal{W}(\Omega)} \left\|D^{\alpha} f(z)\right\|^p_{L^p(S)} \left(\sum_m 2^{nm} 2^{-m(n-1)} \kappa(2^m)\right) \\
&\leq C \sum_{S\in\mathcal{W}(\Omega)} \left\|D^{\alpha} f(z)\right\|^p_{L^p(S)}
\end{aligned}$$

$$= C\left\|D^{\alpha} f(z)\right\|_{L^p(\Omega)}^p$$

where in the penultimate step we used the bound from Lemma 5.3.2. This verifies (5.13) and proves Lemma 5.3.4. □

Estimates for Pairs of Adjacent Cubes

The estimate needed to prove compatibility of the extensions for pairs of adjacent cubes is as follows.

Lemma 5.3.5. *If $\mathcal{E}_Q$ and $\mathcal{E}_{Q'}$ are the operators defined by* (5.8) *for two adjacent cubes Q and Q' then for $1 \leq p < \infty$*

$$\sum_{Q' \in \mathcal{W}} \sum_{Q \in \mathcal{N}(Q')} \sum_{0 \leq \beta \leq \alpha} c(|\alpha - \beta|)^p l(Q')^{-|\alpha-\beta|p} \left\|D^{\beta}(\mathcal{E}_Q f - \mathcal{E}_{Q'} f)\right\|_{L^p(Q' \cap (17/16)Q)}^p \\ \leq C(n, \epsilon, \delta, k, p)\left\|\nabla^k f(y)\right\|_{L^p(\Omega)}^p \tag{5.16}$$

and for $p = \infty$ we have for $x \in Q'$

$$l(Q')^{-|\alpha-\beta|}\left|D^{\beta}(\mathcal{E}_Q f(x) - \mathcal{E}_{Q'} f(x))\right| \leq C\, l(Q')^{k-|\alpha|}\|\nabla^k f\|_{L^{\infty}(\Omega)} \tag{5.17}$$

As this bound is considerably more complicated to obtain than that in Lemma 5.3.4 we begin with a short overview of the method. In essence the plan is as follows. Corresponding to the cubes Q and Q' we have twisting cones Γ_Q and $\Gamma_{Q'}$. We approximate f by degree $(k-1)$ polynomials, using P_Q on the initial piece of Γ_Q and $P_{Q'}$ on the initial piece of $\Gamma_{Q'}$. The difference $f - P_Q$ at any point of the twisting cone Γ_Q will then be controlled by the integral of $|\nabla^k f|$ along Γ_Q as in Lemma 3.4.2, and the polynomial growth of this error term will be dominated by the exponential decay of $\tilde{K}_Q$. Similar estimates will hold for

$f - P_{Q'}$. The polynomials P_Q and $P_{Q'}$ will be invariant under the operator $\mathcal{E}_Q$ as it involves convolution against the reproducing kernel $\tilde{K}_Q$. It will also be possible to show that the difference between P_Q and $P_{Q'}$ is controlled by the integral of $|\nabla^k f|$ along a tube joining the intial pieces of the cones. Combining these estimates will give the bound (5.16).

Proof. It is useful to recogize immediately that it suffices to assume both Q and Q' are in $\mathcal{W}_1$. This is clear if both Q and Q' are too large to be in $\mathcal{W}_1$, since in this instance $\mathcal{E}_Q f \equiv 0 \equiv \mathcal{E}_{Q'} f$ by definition. However the same occurs when only one of the cubes $Q' \in \mathcal{W}_1$, because by the definition of $\mathcal{W}_1$ and (1.2) the neighboring cube has $l(Q) \geq \epsilon\delta/50n$, whence (3.3) shows that the smallest cube in the chain $\{S_j\}$ covering Γ_Q has length at least $2\epsilon\delta/(25\sqrt{n})$ and by (5.6) we know that $f \equiv 0$ on Γ_Q. This again implies $\mathcal{E}_Q f \equiv 0 \equiv \mathcal{E}_{Q'} f$, so all estimates are trivial unless both Q and Q' are in $\mathcal{W}_1$.

To ease readability of the proof we begin here by writing f according to its polynomial approximations on Γ_Q and $\Gamma_{Q'}$, but give estimates for the two different types of terms separately. These appear as Lemma 5.3.6 and Lemma 5.3.8 below.

First we need a a little notation. Recall that the twisting cone Γ_Q corresponding to Q has a central curve γ_Q and at each $z \in \gamma_Q$ a radius $s(z)$. The initial point of γ is called z_0 and the ball B_0 is $B_0 = B(z_0, s(z_0))$. Analogous definitions are made for γ',z_0', and B_0'. In Section 3.4 we defined the polynomial fitted to a function on a set; here we let P_Q be the degree $(k-1)$ polynomial fitted to f on B_0 and $P_{Q'}$ be the corresponding polynomial for f on B_0', so that for any $|\alpha| \leq k$ a multi-index

$$\int_{B_0} D^\alpha (f - P_Q)(x)\, dx = 0 \tag{5.18}$$

$$\int_{B_0'} D^\alpha (f - P_{Q'})(x)\, dx = 0 \tag{5.19}$$

we then wish to rewrite the terms in (5.16) using the expansion

$$
\begin{aligned}
&\mathcal{E}_Q f(x) - \mathcal{E}_{Q'} f(x) \\
&\quad = \int_{\mathbb{R}^n} f_Q(x + l(Q)\tilde{y})\tilde{K}_Q(\tilde{y})\, d\tilde{y} - \int_{\mathbb{R}^n} f_{Q'}(x + l(Q')\tilde{z})\tilde{K}_{Q'}(\tilde{z})\, d\tilde{z} \\
&\quad = \int_{\mathbb{R}^n} (f_Q - P_Q)(x + l(Q)\tilde{y})\tilde{K}_Q(\tilde{y})\, d\tilde{y} + \int_{\mathbb{R}^n} P_Q(x + l(Q)\tilde{y})\tilde{K}_Q(\tilde{y})\, d\tilde{y} \\
&\qquad - \int_{\mathbb{R}^n} (f_{Q'} - P_{Q'})(x + l(Q')\tilde{z})\tilde{K}_{Q'}(\tilde{z})\, d\tilde{z} - \int_{\mathbb{R}^n} P_{Q'}(x + l(Q')\tilde{z})\tilde{K}_{Q'}(\tilde{z})\, d\tilde{z}
\end{aligned} \tag{5.20}
$$

however expressions of this type rapidly become large and unwieldy. We therefore introduce yet another piece of notation. Convolution with the scaling parameter $l(Q)$ will be denoted

$$
g * \tilde{K}_Q(x) = \int_{\mathbb{R}^n} g(x + l(Q)\tilde{y})\tilde{K}_Q(\tilde{y})\, d\tilde{y} \tag{5.21}
$$

so that we may rewrite (5.20) as

$$
\mathcal{E}_Q f(x) - \mathcal{E}_{Q'} f(x) \tag{5.22}
$$

$$
= ((f_Q - P_Q) * \tilde{K}_Q) + (P_Q * \tilde{K}_Q) - (P_{Q'} * \tilde{K}_{Q'}) - ((f_{Q'} - P_{Q'}) * \tilde{K}_{Q'}) \tag{5.23}
$$

If $1 \leq p < \infty$ we take the derivative D^β, the p-th power, and the integral over $(Q' \cap (17/16)Q)$. Using the fact that there are only three terms in the sum we have

$$
\begin{aligned}
&\left\| D^\beta(\mathcal{E}_Q f - \mathcal{E}_{Q'} f) \right\|^p_{L^p(Q' \cap (17/16)Q)} \\
&\leq C(p) \left\| D^\beta((f_Q - P_Q) * \tilde{K}_Q) \right\|^p_{L^p((17/16)Q)} + C(p) \left\| D^\beta((f_{Q'} - P_{Q'}) * \tilde{K}_{Q'}) \right\|^p_{L^p(Q')} \\
&\quad + C(p) \left\| D^\beta(P_Q * \tilde{K}_Q - P_{Q'} * \tilde{K}_{Q'}) \right\|^p_{L^p(Q')}
\end{aligned}
$$

whereupon substituting the bounds from Lemma 5.3.6 and Lemma 5.3.8 completes the proof in the case $1 \leq p < \infty$.

When $p = \infty$ we instead obtain the conclusion directly from (5.22) and the L^∞ estimates of Lemma 5.3.6 and Lemma 5.3.8. □

Polynomial Terms

Lemma 5.3.6. *Let Q and Q' be cubes from $\mathcal{W}_1$, the operators $\mathcal{E}_Q$ and $\mathcal{E}_{Q'}$ be defined as in* (5.8)*, and P_Q and $P_{Q'}$ be the polynomials fitted to f on Q and Q' as described in the discussion preceeding* (5.18)*. Using the notation* (5.21) *we have for $1 \leq p < \infty$*

$$\sum_{Q' \in \mathcal{W}_1} \sum_{Q \in \mathcal{N}(Q')} \sum_{0 \leq \beta \leq \alpha} l(Q')^{-|\alpha-\beta|p} \left\| D^\beta (P_Q * \tilde{K}_Q - P_{Q'} * \tilde{K}_{Q'}) \right\|^p_{L^p(Q')} \leq C \left\| \nabla^k f(y) \right\|^p_{L^p(\Omega)}$$

and for $p = \infty$

$$l(Q')^{-|\alpha-\beta|} \left\| D^\beta (P_Q * \tilde{K}_Q - P_{Q'} * \tilde{K}_{Q'}) \right\|_{L^\infty(Q')} \leq C \left\| \nabla^k f(y) \right\|_{L^\infty(\Omega)} l(Q')^{k-|\alpha|}$$

where $C = C(n, \epsilon, \delta, k, p)$.

We remark that the proof of Lemma 5.3.6 is entirely equivalent to that of Lemma 3.2 of [Jon81]. The only difference is that we derive (5.24) and (5.25) from the properties of the kernel $\tilde{K}_Q$, whereas in [Jon81] this is (essentially) the definition of the operator $\mathcal{E}$.

In the course of the proof we will have occasion to use the following elementary consequence of the fact that any two norms on a finite dimensional Banach space are equivalent.

Lemma 5.3.7. *If $A_1 \subset A_2$ has measure $|A_1| \geq C_1 |A_2|$ then for all $1 \leq p \leq \infty$ there is a uniform bound*

$$||P||_{L^p(A_2)} \leq C(k, C_1) ||P||_{L^p(A_2)}$$

for all polynomials P of degree k.

Proof. Our first observation is that

$$P_Q * \tilde{K}_Q(x) = \int_{\mathbb{R}^n} P_Q(x + l(Q)\tilde{y})\tilde{K}_Q(\tilde{y})\, d\tilde{y}$$

$$= P_Q(x) \tag{5.24}$$

To see this one need only expand the polynomial $P_Q(x + l(Q)\tilde{y})$ as a polynomial in $l(Q)\tilde{y}$ and use the property (5.2) of the kernel $\tilde{K}_Q$. Similarly

$$P_{Q'} * \tilde{K}_{Q'}(x) = P_{Q'}(x) \tag{5.25}$$

It therefore suffices to estimate terms of the form

$$\left\| D^\beta(P_Q - P_{Q'}) \right\|_{L^p(Q')}$$

From (3.3), (3.4), and the definition of B_0' we see that B_0' has diameter comparable to both $l(Q')$ and $\mathrm{dist}(Q', B_0')$. Together with Lemmma 5.3.7 this produces the bound

$$\left\| D^\beta(P_Q - P_{Q'}) \right\|_{L^p(Q')} \leq C \left\| D^\beta(P_Q - P_{Q'}) \right\|_{L^p(B_0')} \tag{5.26}$$

with a constant depending only on n, ϵ and δ.

To estimate (5.26) we use the Poincaré estimate (3.12) and write

$$\begin{aligned} \left\| D^\beta(P_Q - P_{Q'}) \right\|_{L^p(B_0')} &\leq \left\| D^\beta(f - P_{Q'}) \right\|_{L^p(B_0')} + \left\| D^\beta(f - P_Q) \right\|_{L^p(B_0')} \\ &\leq C s'(z_0')^{k-|\beta|} \|\nabla^k f\|_{L^p(B_0')} + \left\| D^\beta(f - P_Q) \right\|_{L^p(B_0')} \end{aligned} \tag{5.27}$$

The latter term is estimated using the version of the Taylor estimate proved in Section 3.4 of Chapter 3. The polynomial fitted to $D^\beta f$ on B_0 is precisely $D^\beta P_Q$. Let $\{T_j\}$ be the chain of cubes connecting the centers of the balls B_0 and B_0'. It was shown in Lemma 3.2.1 that all of these satisfy

$$\frac{1}{C} \leq \frac{l(T_j)}{l(Q')} \leq C$$

where $C = C(n, \epsilon, \delta)$, so that restricting to the case $1 \leq p < \infty$ and applying Lemma 3.4.2 we have

$$\begin{aligned}
\left\|D^\beta(f - P_Q)\right\|_{L^p(B_0')} &\leq C\,(l(T_m))^{k-|\beta|-1} \sum_{j=1}^{m} l(T_j) \left(\frac{l(T_m)}{l(T_j)}\right)^{n/p} \left\|\nabla^k f(y)\right\|_{L^p(T_j)} \\
&\leq C\, l(Q')^{k-|\beta|} \sum_{j=1}^{m} \left\|\nabla^k f(y)\right\|_{L^p(T_j)}
\end{aligned}$$

It was also shown in Lemma 3.2.1 that the number of cubes in a chain of this type is bounded by a number depending on n, ϵ and δ. Using Hölder's inequality we then have

$$\left\|D^\beta(f - P_Q)\right\|^p_{L^p(B_0')} \leq C\, l(Q')^{(k-|\beta|)p} \sum_j \left\|\nabla^k f\right\|^p_{L^p(T_j)}$$

where now $C = C(n, \epsilon, \delta, k, p)$. Combining this with (5.27) and using $s'(z_0') \leq C l(Q')$ yields

$$\begin{aligned}
\left\|D^\beta(P_Q - P_{Q'})\right\|^p_{L^p(B_0')} &\leq C s'(z_0')^{(k-|\beta|)p} \|\nabla^k f\|^p_{L^p(B_0')} + C\, l(Q')^{(k-|\beta|)p} \sum_{j=1}^{m} \left\|\nabla^k f(y)\right\|^p_{L^p(T_j)} \\
&\leq C\, l(Q')^{(k-|\beta|)p} \sum_{j=1}^{m} \left\|\nabla^k f(y)\right\|^p_{L^p(T_j)} \qquad (5.28)
\end{aligned}$$

In order to sum terms of the form

$$l(Q')^{-|\alpha-\beta|p} \left\|P_Q * \tilde{K}_Q - P_{Q'} * \tilde{K}_{Q'}\right\|^p_{L^p(Q')}$$

over all $Q' \in \mathcal{W}_1$, $Q \in \mathcal{N}(Q')$ and $0 \leq \beta \leq \alpha$, it is helpful to use the notation introduced in (3.8) of Section 3.3. We defined $\mathcal{F}(Q')$ to be all cubes occuring in chains connecting locations of size comparable to $l(Q')$ and separated from Q' by distance like $l(Q')$. It is apparent that the chain $\{T_m\}$ of (5.28) is of this type with constants depending on ϵ, δ and n, whereupon the estimate (3.9) allows us to calculate

$$\begin{aligned}
&\sum_{Q'\in\mathcal{W}_1}\sum_{Q\in\mathcal{N}(Q)}\sum_{0\leq\beta\leq\alpha} l(Q')^{-|\alpha-\beta|p}\left\|P_Q * \tilde{K}_Q - P_{Q'} * \tilde{K}_{Q'}\right\|^p_{L^p(Q')} \\
&\leq C\sum_{Q'\in\mathcal{W}}\sum_{0\leq\beta\leq\alpha} l(Q')^{-|\alpha-\beta|p} l(Q')^{(k-|\beta|)p}\sum_{T\in\mathcal{F}(Q')}||\nabla^k f||^p_{L^p(T)} \\
&\leq C\sum_{Q'\in\mathcal{W}}\sum_{T\in\mathcal{F}(Q')}||\nabla^k f||^p_{L^p(T)} l(Q')^{(k-|\alpha|)p} \\
&\leq C\sum_{T\in\mathcal{W}(\Omega)}||\nabla^k f||^p_{L^p(T)} \\
&= C\,||\nabla^k f||^p_{L^p(\Omega)}
\end{aligned}$$

where in the second to last inequality we used that $|\alpha|\leq k$ and that there is a bound on the size of cubes $Q'\in\mathcal{W}_1$. This concludes the proof for the case $1\leq p<\infty$.

To complete the proof for $f\in W^{k,\infty}(\Omega)$ we return to (5.27) and use (3.15) of Lemma 3.4.2 to write

$$\begin{aligned}
\left\|D^\beta(P_Q - P_{Q'})\right\|_{L^\infty(B_0')} &\leq \left\|D^\beta(f-P_{Q'})\right\|_{L^\infty(B_0')} + \left\|D^\beta(f-P_Q)\right\|_{L^\infty(B_0')} \\
&\leq Cs'(z_0')^{k-|\beta|}||\nabla^k f||_{L^\infty(B_0')} + C\,l(Q')^{k-|\beta|}||\nabla^k f||_{L^\infty(\Omega)} \\
&\leq\ l(Q')^{k-|\beta|}||\nabla^k f||_{L^\infty(\Omega)}
\end{aligned}$$

because both the diameter of B_0' and the separation of B_0 from B_0' are comparable to $l(Q')$ with constants $C(n,\epsilon,\delta)$. Substituting into (5.26) and multiplying by $l(Q')^{-|\alpha-\beta|}$ gives

$$l(Q')^{-|\alpha-\beta|}\left\|D^\beta(P_Q * \tilde{K}_Q - P_{Q'} * \tilde{K}_{Q'})\right\|_{L^\infty(Q')} \leq C\,||\nabla^k f||_{L^\infty(\Omega)} l(Q')^{k-|\alpha|}$$

□

Terms involving $(\mathbf{f} - \mathbf{P_Q})$

Lemma 5.3.8. *Let Q and Q' be cubes from $\mathcal{W}_1$, the operators $\mathcal{E}_Q$ and $\mathcal{E}_{Q'}$ be defined as in* (5.8)*, and P_Q and $P_{Q'}$ be the polynomials fitted to f on Q and Q' as described in the discussion preceeding* (5.18)*. Using the notation* (5.21) *we have for $1 \leq p < \infty$*

$$\sum_{Q' \in \mathcal{W}_1} \sum_{Q \in \mathcal{N}(Q')} \sum_{0 \leq \beta \leq \alpha} l(Q)^{-|\alpha-\beta|p} \left\| D^\beta((f_Q - P_Q) * \tilde{K}_Q) \right\|^p_{L^p((17/16)Q)} \leq C \left\| \nabla^k f(y) \right\|^p_{L^p(\Omega)} \qquad (5.29)$$

while for $p = \infty$

$$l(Q)^{-|\alpha-\beta|} \left\| D^\beta((f_Q - P_Q) * \tilde{K}_Q) \right\|_{L^\infty((17/16)Q)} \leq C \left\| \nabla^k f(y) \right\|_{L^\infty(\Omega)} l(Q)^{k-|\alpha|} \qquad (5.30)$$

where $C = C(n, \epsilon, \delta, k, p)$.

Proof. We first differentiate within the integral to write

$$D^\beta((f_Q - P_Q) * \tilde{K}_Q)(x) = \int_{\mathbb{R}^n} D^\beta(f_Q - P_Q)(x + l(Q)\tilde{y})\tilde{K}_Q(\tilde{y})\, d\tilde{y}$$

as in (5.14) and make the change of variables $z = (x + l(Q)\tilde{y})$ to obtain

$$D^\beta((f_Q - P_Q) * \tilde{K}_Q)(x) = \frac{1}{l(Q)^n} \int_{\mathbb{R}^n} D^\beta(f_Q - P_Q)(z)\tilde{K}_Q\left(\frac{z - x}{l(Q)}\right) dz$$

Now by Lemma 5.2.1 we know that all points at which $\tilde{K}_Q((z - x)/l(Q)) \neq 0$ lie either in the union of cubes S_j from the chain covering Γ_Q, or within distance $\sqrt{n}l(Q)$ of $\Gamma^*_Q \setminus \Gamma_Q$. Moreover $f_Q \equiv 0$ outside $\cup S_j$ and we have the bound (5.4) for $\tilde{K}_Q$ when $z \in S_j$ and

$x \in (17/16)Q$. This allows us to write

$$\begin{aligned}\left|D^{\beta}((f_Q - P_Q) * \tilde{K}_Q)(x)\right| \leq \sum_j \left(\frac{l(Q)}{l(S_j)}\right)^{n-1} \kappa\left(\frac{l(S_j)}{l(Q)}\right) \int_{S_j} \left|D^{\beta}(f_Q - P_Q)(z)\right| \frac{dz}{l(Q)^n} \\ + \int_{\tilde{\Gamma}^*_Q \setminus \tilde{\Gamma}_Q} \left|D^{\beta} P_Q(x + l(Q)\tilde{y})\right| \left|\tilde{K}_Q(\tilde{y})\right| d\tilde{y}\end{aligned}$$

It is possible to write a similar estimate for the term involving the integral over $\tilde{\Gamma}^*_Q \setminus \tilde{\Gamma}_Q$. All we need do is define a collection $\{T_m\}$ of cubes such that each T_m has length comparable to its separation from Q and so $\cup T_m$ contains all points within distance $\sqrt{n} l(Q)$ of $\Gamma^*_Q \setminus \Gamma_Q$. This is clearly possible from the definition of $\tilde{\Gamma}^*_Q$ and we see that all of the constants of comparability depend on n, ϵ, and δ. In particular it is evident that (5.4) is still valid for these new cubes. We may then adjoin $\{T_m\}$ to the chain $\{S_j\}$ so that we have a chain covering all of Γ^*_Q. Not all cubes in the chain are Whitney cubes of Ω, but we need only keep in mind that $f_Q \equiv 0$ on all those that are not. Using this convention we obtain

$$\left|D^{\beta}((f_Q - P_Q) * \tilde{K}_Q)(x)\right| \leq \sum_j \left(\frac{l(Q)}{l(S_j)}\right)^{n-1} \kappa\left(\frac{l(S_j)}{l(Q)}\right) \int_{S_j} \left|D^{\beta}(f_Q - P_Q)(z)\right| \frac{dz}{l(Q)^n} \tag{5.31}$$

Now suppose $1 \leq p < \infty$ and apply (3.14) of Lemma 3.4.2 with the exponent $p = 1$ to the integrals. This gives

$$\int_{S_j} \left|D^{\beta}(f_Q - P_Q)(z)\right| dz \leq C\,(l(S_j))^{k-|\beta|-1} \sum_{m=1}^{j} l(S_m) \left(\frac{l(S_j)}{l(S_m)}\right)^n \left\|\nabla^k f(y)\right\|_{L^1(S_m)}$$

so that

$$\int_{S_j} \left|D^{\beta}(f_Q - P_Q)(z)\right| \frac{dz}{l(Q)^n} \leq C\,(l(S_j))^{k-|\beta|-1} \left(\frac{l(S_j)}{l(Q)}\right)^n \sum_{m=1}^{j} l(S_m) \fint_{S_m} \left|\nabla^k f_Q(y)\right| dy$$

This is even valid on the cubes that we appended to the chain; we keep in mind that $f_Q \equiv 0$

on those cubes. Substituting back into (5.31)

$$\begin{aligned}
&\left|D^\beta((f_Q - P_Q) * \tilde{K}_Q)(x)\right| \\
&\qquad \le C \sum_j \left(\frac{l(S_j)}{l(Q)}\right) \kappa\left(\frac{l(S_j)}{l(Q)}\right) (l(S_j))^{k-|\beta|-1} \sum_{m=1}^{j} l(S_m) \fint_{S_m} \left|\nabla^k f_Q(y)\right| dy \\
&\qquad = C\, l(Q)^{k-|\beta|} \sum_j \left(\frac{l(S_j)}{l(Q)}\right)^{k-|\beta|} \kappa\left(\frac{l(S_j)}{l(Q)}\right) \sum_{m=1}^{j} \frac{l(S_m)}{l(Q)} \fint_{S_m} \left|\nabla^k f_Q(y)\right| dy \\
&\qquad = C\, l(Q)^{k-|\beta|} \sum_m \frac{l(S_m)}{l(Q)} \fint_{S_m} \left|\nabla^k f_Q(y)\right| dy \left[\sum_{j=m}^{\infty} \left(\frac{l(S_j)}{l(Q)}\right)^{k-|\beta|} \kappa\left(\frac{l(S_j)}{l(Q)}\right)\right]
\end{aligned}$$

however the number of S_j of a given scale is bounded by constants depending on n, ϵ and δ, so applying Lemma 5.3.2

$$\sum_{j=m}^{\infty} \left(\frac{l(S_j)}{l(Q)}\right)^{k-|\beta|} \kappa\left(\frac{l(S_j)}{l(Q)}\right) \le C(n,\epsilon,\delta,k) \left(\frac{l(S_m)}{l(Q)}\right)^{k-|\beta|} \kappa\left(\frac{l(S_m)}{l(Q)}\right)$$

and hence

$$\left|D^\beta((f_Q - P_Q) * \tilde{K}_Q)(x)\right| \le C\, l(Q)^{k-|\beta|} \sum_m \left(\frac{l(S_m)}{l(Q)}\right)^{k-|\beta|+1} \kappa\left(\frac{l(S_m)}{l(Q)}\right) \fint_{S_m} \left|\nabla^k f_Q(y)\right| dy$$

Taking the p-th power we may use Hölder's inequality, then the estimate from Lemma 5.3.2 with $q = (kp - |\beta|p + p - n)/(p-1)$, and then Jensen's inequality to conclude

$$\begin{aligned}
&\left|D^\beta((f_Q - P_Q) * \tilde{K}_Q)(x)\right|^p \\
&\le C\, l(Q)^{(k-|\beta|)p} \left[\sum_{m=1}^{\infty} \left(\frac{l(S_m)}{l(Q)}\right)^n \kappa\left(\frac{l(S_m)}{l(Q)}\right) \left(\fint_{S_m} |\nabla^k f_Q(y)|\, dy\right)^p\right] \left[\sum_{m=1}^{\infty} \left(\frac{l(S_m)}{l(Q)}\right)^q \kappa\left(\frac{l(S_m)}{l(Q)}\right)\right]^{p-1} \\
&\le C\, l(Q)^{(k-|\beta|)p} \sum_{m=1}^{\infty} \left(\frac{l(S_m)}{l(Q)}\right)^n \kappa\left(\frac{l(S_m)}{l(Q)}\right) \fint_{S_m} |\nabla^k f_Q(y)|^p\, dy \\
&\le C l(Q)^{(k-|\beta|)p-n} \sum_{m=1}^{\infty} \kappa\left(\frac{l(S_m)}{l(Q)}\right) \int_{S_m} |\nabla^k f_Q(y)|^p\, dy
\end{aligned}$$

As the estimate is independent of x, integration over $(17/16)Q$ merely increases the constant marginally and cancels a factor of $l(Q)^{-n}$. We then have

$$\left\|D^{\beta}((f_Q - P_Q) * \tilde{K}_Q)(x)\right\|^p_{L^p((17/16)Q)} \leq Cl(Q)^{(k-|\beta|)p} \sum_{m=1}^{\infty} \kappa\left(\frac{l(S_m)}{l(Q)}\right) \int_{S_m} |\nabla^k f_Q(y)|^p \, dy \quad (5.32)$$

which is at last in a form appropriate for estimating the sum in (5.29). If we multiply (5.32) by $l(Q)^{-|\alpha-\beta|p}$ and sum as in (5.29) we obtain

$$\begin{aligned}\sum_{Q'\in\mathcal{W}_1} \sum_{Q\in\mathcal{N}(Q')} \sum_{0\leq\beta\leq\alpha} l(Q)^{-|\alpha-\beta|p} \left\|D^{\beta}((f_Q - P_Q) * \tilde{K}_Q)\right\|^p_{L^p((17/16)Q)} \\ \leq C \sum_{Q'\in\mathcal{W}_1} \sum_{Q\in\mathcal{N}(Q')} \sum_{0\leq\beta\leq\alpha} l(Q)^{(k-|\alpha|)p} \sum_{S_m\cap\Gamma^*_Q\neq\emptyset} \kappa\left(\frac{l(S_m)}{l(Q)}\right) \int_{S_m} |\nabla^k f_Q(y)|^p \, dy\end{aligned}$$

but a cube Q' has a bounded number of neighbors $Q \in \mathcal{N}(Q')$ and there are at most $C(n, k)$ values of β with $0 \leq \beta \leq \alpha$ and $|\alpha| \leq k$. Moreover $Q \in \mathcal{W}_1$ has $l(Q) \leq C(n, \epsilon, \delta)$ so $|\alpha| \leq k$ implies $l(Q)^{(k-|\alpha|)p} \leq 1$. If we write $\mathcal{W}_2$ for the collection of cubes that are neighbors of cubes from $\mathcal{W}_1$ the estimate then reduces to

$$\begin{aligned}\sum_{Q'\in\mathcal{W}_1} \sum_{Q\in\mathcal{N}(Q')} \sum_{0\leq\beta\leq\alpha} l(Q)^{-|\alpha-\beta|p} \left\|D^{\beta}((f_Q - P_Q) * \tilde{K}_Q)\right\|^p_{L^p((17/16)Q)} \\ \leq C \sum_{Q\in\mathcal{W}_2} \sum_{S_m\cap\Gamma^*_Q\neq\emptyset} \kappa\left(\frac{l(S_m)}{l(Q)}\right) \int_{S_m} |\nabla^k f_Q(y)|^p \, dy\end{aligned}$$

Note that since $f_Q \equiv 0$ on the cubes S_j that do not intersect Γ_Q we may leave those out of the inner sum. The cubes that remain are Whitney cubes of Ω on which $f_Q \equiv f$. Reversing the order of summation and using the notation of Section 3.3 they may be written as $Q \in \mathcal{G}(S)$. It was proven in (3.10) that the number of these cubes having scale $2^{-j} l(S)$ is bounded by a constant multiple of 2^{nj}, so

$$\begin{aligned}
\sum_{Q'\in\mathcal{W}_1}\sum_{Q\in\mathcal{N}(Q')}\sum_{0\le\beta\le\alpha} l(Q)^{-|\alpha-\beta|p}\left\|D^\beta((f_Q-P_Q)*\tilde{K}_Q)\right\|^p_{L^p((17/16)Q)} \\
&\le C\sum_{Q\in\mathcal{W}_2}\sum_{S_m\cap\Gamma_Q\neq\emptyset}\kappa\left(\frac{l(S_m)}{l(Q)}\right)\int_{S_m}|\nabla^k f(y)|^p\,dy \\
&= C\sum_{S\in\mathcal{W}(\Omega)}\int_S|\nabla^k f(y)|^p\,dy\sum_{Q\in\mathcal{G}(S)}\kappa\left(\frac{l(S_m)}{l(Q)}\right) \\
&\le C\sum_{S\in\mathcal{W}(\Omega)}\int_S|\nabla^k f(y)|^p\,dy\sum_{j=0}^{\infty}2^{nj}k(2^j) \\
&\le C\sum_{S\in\mathcal{W}(\Omega)}\int_S|\nabla^k f(y)|^p\,dy \\
&\le C\left\|\nabla^k f(y)\right\|^p_{L^p(\Omega)}
\end{aligned}$$

where the penultimate estimate is from Lemma 5.3.2.

As has been true throughout, the proof is easier in the case $p=\infty$. Returning to (5.31) we need only use (3.15) of Lemma 3.4.2 to deduce

$$\begin{aligned}
\left|D^\beta((f_Q-P_Q)*\tilde{K}_Q)(x)\right| &\le \|\nabla^k f\|_{L^\infty(\Omega)}\sum_j\left(\frac{l(S_j)}{l(Q)}\right)l(S_j)^{k-|\beta|}\kappa\left(\frac{l(S_j)}{l(Q)}\right) \\
&\le C\,l(Q)^{k-|\beta|}\|\nabla^k f\|_{L^\infty(\Omega)}\sum_j\left(\frac{l(S_j)}{l(Q)}\right)^{k-|\beta|+1}\kappa\left(\frac{l(S_j)}{l(Q)}\right) \\
&\le C\,l(Q)^{k-|\beta|}\|\nabla^k f\|_{L^\infty(\Omega)}
\end{aligned}$$

where we used the fact that only finitely many S_j of a given scale intersect the twisting cone, and the estimate from Lemma 5.3.2. Multiplying by $l(Q)^{-|\alpha-\beta|}$ gives the desired result. □

5.4 Completing the Proof

For $f \in W^{k,p}(\Omega)$ we have now defined $\mathcal{E}f$ on all but $\partial\Omega$, which is a set of measure zero, and we know that the $W^{k,p}$ norm of $\mathcal{E}f$ on both Ω and $(\Omega^c)^o$ is controlled by $||f||_{W^{k,p}(\Omega)}$. All that remains is to see that $f \in W^{k,p}(\mathbb{R}^n)$. This may be thought of as checking that $\mathcal{E}f$ on $(\Omega^c)^o$ "joins up" correctly with f at $\partial\Omega$. The situation in which this is most readily proved is when f is a smooth function on $\mathbb{R}^n$ with bounded derivatives on Ω, and we can reduce to this case using the following result of Jones (Proposition 4.4 of [Jon81]).

Proposition 5.4.1 (Jones). *For fixed $\eta > 0$, k, $p \in [1,\infty)$, and $f \in W^{k,p}(\Omega)$ there is $g \in C^\infty(\mathbb{R}^n) \cap W^{k,p}(\Omega)$ and $M \in \mathbb{R}$ with*

$$\begin{aligned} ||f-g||_{W^{k,p}(\Omega)} &\leq C\eta \\ |D^\alpha g| \leq M \quad &\textit{for } 0 \leq |\alpha| \leq k \end{aligned} \tag{5.33}$$

while for fixed $f \in W^{k,\infty}(\Omega)$ there is $g \in C^\infty\mathbb{R}^n \cap W^{k,\infty}(\Omega)$ with

$$\begin{aligned} ||f-g||_{W^{k-1,\infty}(\Omega)} &\leq C\eta \\ ||g||_{W^{k,\infty}(\Omega)} &\leq C||f||_{W^{k,\infty}(\Omega)} \end{aligned} \tag{5.34}$$

Proof. We give only a sketch of the proof; further details may be found in the original work [Jon81]. Note first that the usual methods of mollification on Lipschitz domains do not work everywhere on locally uniform domains, but at distance d from $\partial\Omega$ it is perfectly legitimate to mollify using a smooth bump function supported on a ball of radius $d/2$. The difficulties in the proof involve what can be done near $\partial\Omega$.

Jones uses the following procedure to obtain a smooth approximation to the function in a neighborhood of the boundary. First he takes a collection of Whitney cubes that are neither too small nor too large and divides them up to reach a particular dyadic scale. These should be thought of as forming a thin band parallel to the length of the boundary. To each of these cubes he associates the polynomial fitted to the cube as discussed in Section 3.4 of Chapter 3, so that it matches f and its derivatives of order less than k on the cube. Then he magnifies all of these cubes by a sufficiently large factor that their union covers a neighborhood of $\partial\Omega$. The neighborhood has some known size depending on n, ϵ and δ. On the magnified cubes he takes smooth partition of unity which is used to smoothly sum the fitted polynomials, in each case multiplying the polynomial for a cube by the smooth bump function for the corresponding magnified cube. The proof that this gives a smooth approximation to the function near $\partial\Omega$ involves joining pairs of cubes by tubes of the type discussed in Section 3.3 and bounding the variation of the polynomials by the integral of $|\nabla^k f|$ along the tube in a manner akin to the proof of Lemma 5.3.6.

Once he has a smooth function to use near the boundary, Jones takes a smooth cutoff function χ (as in Section 5.1) and uses χ and $(1-\chi)$ to divide Ω into a narrow neighborhood of the boundary and a region well separated from the boundary. The width of this neighborhood is chosen so that the $W^{k,p}$ norm of f on the narrow neighborhood of $\partial\Omega$ is bounded by η, and the approximation to f near the boundary is defined to be χ times the polynomial approximation discussed above. The remaining region is some fixed distance from the boundary, so f is smoothed using standard mollifier supported on balls that remain away from $\partial\Omega$ before being multiplied by $(1-\chi)$ to give the second piece of the approximating function. $\square$

For smooth functions of this type it is not difficult to prove that applying the extension operator produces a function for which all derivatives of orders less than k are Lipschitz at small scales. We record this as a lemma.

Lemma 5.4.2. *Fix $k \in \mathbb{N}$ and $p \in [1,\infty]$ and let $g \in W^{k,p}(\Omega)$ satisfy the conclusions of*

Proposition 5.4.1. Then for any $0 \leq |\alpha| < k$ *the function* $D^\alpha \mathcal{E}g$ *is locally Lipschitz on* $\mathbb{R}^n$.

Proof. Fix α with $0 \leq |\alpha| < k$. If $x \in \Omega$ then $\mathcal{E}g = g$ in a neighborhood of x and $D^\alpha g$ is Lipschitz there by the appropriate choice of (5.33) or (5.34). Moreover it follows from the L^∞ case of Theorem 5.3.1 that $\mathcal{E}g$ satisfies the same bounds (with a multiplicative constant) on $(\Omega^c)^o$ and is therefore Lipschitz in a neighborhood of $x \in (\Omega^c)^o$ by the same argument. We therefore need only show that $D^\alpha g$ is Lipschitz in a neighborhood of any point of $\partial\Omega$, for which purpose it clearly suffices that there is a constant $s > 0$ such that if $x \in (\Omega^c)^o$ and $y \in \Omega$ with $|x - y| < s$ then

$$\left|D^\alpha(\mathcal{E}g(x) - \mathcal{E}g(y))\right| \leq C|x - y| \tag{5.35}$$

We will take $s = \epsilon\delta/200n$. Fix $x \in (\Omega^c)^o$ and $y \in \Omega$ with $|x - y| < s$. Let Q be the Whitney cube of $(\Omega^c)^o$ that contains x, let x_Q denote the center of Q, and take y_Q to be the initial point of the curve γ around which we have the twisting cone Γ_Q. Integration against $\tilde{K}_Q$ preserves polynomials, so in particular it will preserve the constant function $L = D^\alpha g(y_Q)$. Since $\mathcal{E}g(x_Q) = \mathcal{E}_Q g(x_Q)$ we may compute

$$\begin{aligned}
\left|D^\alpha \mathcal{E}g(x_Q) - D^\alpha g(y_Q)\right| &= \left|\int_{\mathbb{R}^n} (D^\alpha g_Q(x_Q + l(Q)\tilde{y}) - L)\tilde{K}_Q(\tilde{y})\, d\tilde{y}\right| \\
&\leq \int_{\mathbb{R}^n} \left|D^\alpha g_Q(x_Q + l(Q)\tilde{y}) - L\right|\left|\tilde{K}_Q(\tilde{y})\right| d\tilde{y}
\end{aligned}$$

Reasoning as in the proof of the L^∞ estimate for Lemma 5.3.8 we see that

$$\begin{aligned}
\left|D^\alpha g_Q(x + l(Q)\tilde{y}) - L\right| &= \left|D^\alpha g_Q(x_Q + l(Q)\tilde{y}) - D^\alpha g(y_Q)\right| \\
&\leq C\left|x_Q + l(Q)\tilde{y} - y_Q\right|^{k-|\alpha|} \|\nabla^k g\|_{L^\infty(\Omega)}
\end{aligned}$$

and this may be integrated against $|\tilde{K}_Q|$ to provide

$$\begin{aligned}\left|D^\alpha \mathcal{E}g(x_Q) - D^\alpha g(y_Q)\right| &\le C\, l(Q)^{k-|\alpha|} \|\nabla^k g\|_{L^\infty(\Omega)} \\ &\le C\, |x-y|^{k-|\alpha|} \|\nabla^k g\|_{L^\infty(\Omega)} \end{aligned} \tag{5.36}$$

We also know from Lemma 3.2.3

$$|x - x_Q| \le \operatorname{dist}(x_Q, \Omega) \le |x-y| \tag{5.37}$$

$$|x_Q - y_Q| \le 20\sqrt{n}\, l(Q) \le C|x-y| \tag{5.38}$$

It follows from (5.37) and the known bound on $|D^\alpha \mathcal{E}g|_{L^\infty((\Omega^c)^o)}$ that

$$\left|\mathcal{E}g(x) - \mathcal{E}g(x_Q)\right| \le C \|\nabla^k g\|_{L^\infty(\Omega)} |x-y| \tag{5.39}$$

and from (5.38) that $|y_Q - y| \le 25\sqrt{n}|x-y|$. This is certainly less than δ so we may connect y to y_Q with a chain of cubes and apply the L^∞ estimate in Lemma 3.4.2 to conclude

$$\left|D^\alpha g(y) - D^\alpha g(y_Q)\right| \le C \|\nabla^k g\|_{L^\infty(\Omega)} |x-y|^{k-|\alpha|}$$

This may be combined with (5.36), (5.39), and the fact $|x-y| < 1$ to prove (5.35). □

Using Lemma 5.4.2 we see that any g satisfying the conclusions of Proposition 5.4.1 has locally Lipschitz derivatives of all orders less than k and is therefore k-times differentiable almost everywhere. As $\partial\Omega$ has measure zero we conclude from Theorem 5.3.1 that $g \in W^{k,p}(\mathbb{R}^n)$ and

$$\|\mathcal{E}g\|_{W^{k,p}(\mathbb{R}^n)} \le C \|g\|_{W^{k,p}(\Omega)}$$

so that $\mathcal{E}$ is a bounded linear operator on this space of functions. Proposition 5.4.1 shows that we can approximate (or weakly approximate in the case $p = \infty$) any $f \in W^{k,p}(\Omega)$ by such g, and consequently that $\mathcal{E}f$ is in $W^{k,p}(\mathbb{R}^n)$ and satisfies the same estimate. This completes the proof of Theorem 2.1.1.

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