Degree-Independent Sobolev Extension on Locally Uniform Domains

Luke G. Rogers

Department of Mathematics, Cornell University, Ithaca, NY 14850, USA

Abstract

We consider the problem of constructing extensions $L_k^p(\Omega) \to L_k^p(\mathbb{R}^n)$, where L_k^p is the Sobolev space of functions with k derivatives in L^p and $\Omega \subset \mathbb{R}^n$ is a domain. In the case of Lipschitz Ω, Calderón gave a family of extension operators depending on k, while Stein later produced a single (k-independent) operator. For the more general class of locally-uniform domains, which includes examples with highly non-rectifiable boundaries, a k-dependent family of operators was constructed by Jones. In this work we produce a k-independent operator for all spaces $L_k^p(\Omega)$ on a locally uniform domain Ω.

Key words: Sobolev extension, locally uniform domain
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1 Introduction

We work on the Euclidean space $\mathbb{R}^n$ of dimension $n \geq 2$, and on a connected open domain Ω. Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ be a multi-index with length $|\alpha| = \sum \alpha_j$. Suppose f and g are locally integrable on Ω and are related by the integration by parts formula

$$\int_\Omega f(x)(D^\alpha \phi(x))\, dx = (-1)^{|\alpha|} \int_\Omega g(x)\phi(x)\, dx$$

for all $\phi \in C^\infty$ with compact support in Ω, where $D^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$. Then we call g the weak derivative of f of order α, and write $g = D^\alpha f$.

The Sobolev space $L_k^p(\Omega)$ consists of those locally integrable functions f which have weak derivatives in $L^p(\Omega)$ for all α with $|\alpha| \leq k$. It is a Banach space with

Email address: `luke@math.cornell.edu` (Luke G. Rogers).

norm

$$\|f\|_{L^p_k(\Omega)} = \sum_{|\alpha|\leq k} \|D^\alpha f\|_{L^p(\Omega)}.$$

If we compare the Sobolev spaces on $\mathbb{R}^n$ to those on a subdomain Ω, it is evident that there is a bounded linear mapping $L^p_k(\mathbb{R}^n) \to L^p_k(\Omega)$ induced by the restriction $f \mapsto f|_\Omega$. This paper is a contribution to the ongoing work of many authors on the corresponding extension problem (see [8,3,2,16,5,11,12,6,21,15]), which may be briefly summarized as follows.

Question 1 *What may be said about the existence and properties of bounded linear extension mappings $E : L^p_k(\Omega) \to L^p_k(\mathbb{R}^n)$ with $Ef|_\Omega = f$?*

A simple example shows that this problem depends non-trivially on the geometry of $\partial\Omega$.

Example 2 *Consider $f(x,y) = x^{-a}$ on $\Omega = \{(x,y) \in \mathbb{R}^2 : |y| < x^b, x \in (-1,1)\}$ with $b > 1$. For $a > 0$ and $\epsilon > 0$ so small that $b-(a+1)(2+\epsilon) > -1$ we have $f \in L^{2+\epsilon}_1$, but this has no extension in $L^{2+\epsilon}_1$ as the Sobolev embedding theorem implies the latter is a space of Hölder continuous functions.*

Extension on Lipschitz Domains

In view of the obstruction posed by a cusp on $\partial\Omega$ it is perhaps unsurprising that the classical affirmative results are for Lipschitz domains. The following theorem of Calderón [2] was the first to deal with general orders of smoothness k, and was later improved by Stein [17,18] using an entirely different proof.

Theorem 3 (Calderón) *Let $\Omega \subset \mathbb{R}^n$ be Lipschitz. For each $k \in \mathbb{N}$ there is a bounded linear extension operator such that for all $1 < p < \infty$*

$$E_k : L^p_k(\Omega) \longrightarrow L^p_k(\mathbb{R}^n)$$

with bound depending on n, k, p and the constants of the Lipschitz domain.

Theorem 4 (Stein) *Let $\Omega \subset \mathbb{R}^n$ be Lipschitz. There is a bounded linear extension operator such that for any $k \in \mathbb{N}$ and $1 \leq p \leq \infty$*

$$E : L^p_k(\Omega) \longrightarrow L^p_k(\mathbb{R}^n).$$

with bound depending on n, k, p and the constants of the Lipschitz domain.

Notice that Calderón produces a family of extension operators E_k, one for each order of smoothness. By contrast, Stein constructs a single *degree independent* extension operator. In what follows we shall examine the existence of degree independent operators on a much larger class of domains.

Locally uniform domains were introduced by Martio and Sarvas [13], but the following equivalent definition is from [11].

Definition 5 *A domain is (ϵ, δ) locally uniform if between any pair of points x,y such that $|x - y| < \delta$ there is a rectifiable arc $\gamma \subset \Omega$ of length at most $|x - y|/\epsilon$ and having the property that for all $z \in \gamma$*

$$\operatorname{dist}(z, \partial\Omega) \geq \frac{\epsilon|z - x||z - y|}{|x - y|}. \tag{1}$$

These domains have close connections to quasiconformal mappings [4] and enjoy a wide variety of potential-theoretic properties akin to those of the half-spaces $\mathbb{R}^n_+$ [9]. Unlike Lipschitz domains, they may have highly non-rectifiable boundaries: the boundary of a locally uniform domain in $\mathbb{R}^n$ may have any dimension in $[n - 1, n)$. The extension properties of locally uniform domains were first studied by Jones, who proved that they are precisely the domains on which BMO functions can be extended [10], and that they have the following Sobolev extension properties [11].

Theorem 6 (Jones) *Let $\Omega \subset \mathbb{R}^n$ be an (ϵ, δ) locally uniform domain. For each fixed $k \in \mathbb{N}$ there is a bounded linear extension operator such that for all $1 \leq p \leq \infty$*

$$\mathcal{E}_k : L^p_k(\Omega) \longrightarrow L^p_k(\mathbb{R}^n)$$

with a bound depending on n, ϵ, δ, k and p.

Theorem 7 (Jones) *If $\Omega \subset \mathbb{R}^2$ is bounded and finitely connected then the following are equivalent*

(i) *There are extension operators $\mathcal{E}_k$ as in Theorem 6.*
(ii) *Ω is an (ϵ, ∞) locally uniform domain.*
(iii) *$\partial\Omega$ consists of a finite number of points and quasicircles.*

From these theorems we know both that the locally uniform domains admit Sobolev extension operators and that they are the most general class to do so in $\mathbb{R}^2$. Certain known examples suggest that there is no simple geometric condition like that in Theorem 7 to characterize extension domains in higher dimensions, though some progress has been made by Herron and Koskela [7,6].

One limitation of Jones' results is that the operators $\mathcal{E}_k$ are far from degree independent. In fact $\mathcal{E}_k$ is not even defined on the spaces $L^p_l(\Omega)$ for $l < k$. The purpose of the present paper is to offer an alternative approach to Sobolev extensions on locally uniform domains that results in a degree independent operator.

Theorem 8 *Let $\Omega \subset \mathbb{R}^n$ be an (ϵ, δ) locally uniform domain. There is a linear operator $f \mapsto \mathcal{E}f$ such that for any $k \in \mathbb{N}$ and $1 \leq p \leq \infty$*

$$\mathcal{E} : L^p_k(\Omega) \longrightarrow L^p_k(\mathbb{R}^n) \tag{2}$$

$$\|\mathcal{E}f\|_{L^p_k(\mathbb{R}^n)} \leq c(n, \epsilon, \delta, k, p)\|f\|_{L^p_k(\Omega)}. \tag{3}$$

The proof of Theorem 8 follows the method developed by Whitney for his celebrated Lipschitz extension theorem [22]. We decompose the interior of $\Omega^c = \mathbb{R}^n \setminus \Omega$ into a union of cubes, define an extension for each cube and then sum using a smooth partition of unity. This is the same approach used by Jones in [11] and some of our arguments parallel his, however the proofs differ substantially in the method used to construct an extension corresponding to an individual Whitney cube. To obtain a degree independent extension we need to capture the behavior of f up to arbitrary orders, and this requires quite different techniques than are needed when the order of approximation is fixed in advance. The bulk of this work is found in Section 3 and summarized in Theorem 16. It involves solving a certain moment problem under a geometric constraint on the support of the solution, and was inspired by Stein's use of a corresponding one-dimensional result (Lemma 1 on page 182 of [18]) in his construction for the Lipschitz case.

Before embarking upon the proof we warn the reader that Theorem 8 will be only proved under the additional assumption that Ω has diameter at least 1. This allows us to avoid renormalizing polynomials of degree less than k to have norm zero in L^p_k, an operation which is routine but adds unnecessary technicalities to the proof. As a result the constant c in (3) will grow without bound if the diameter of Ω is sent to zero while all other constants in (3) remain fixed.

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2 Geometry

Points in $\mathbb{R}^n$ are denoted x or $(x_1, x_2, \ldots, x_n)$. The Euclidean distance between two points is $|x - y|$, the distance from x to a set A is $\operatorname{dist}(x, A)$, and the distance between two sets is $\operatorname{dist}(A, B)$. Balls are written $B(x, r) = \{y : |x - y| \leq r\}$. At times it will be convenient to write λB for the ball concentric with B but having λ times its radius.

A set of the form $Q_l(x) = \{y : |y_j - x_j| \leq l/2\}$ is a cube of center x and length l. The center of the cube Q is denoted x_Q and its length is $l(Q)$. As with balls, λQ is the cube with the same center as Q but length λ times as large. A dyadic cube of scale $2^j, j \in \mathbb{Z}$, is a cube having size 2^j and all of whose vertices lie on the lattice $(2^j\mathbb{Z})^n$.

We make extensive use of Whitney's decomposition of an open set into cubes. A proof may be found in Stein [18] Chapter VI, Section 1.

Lemma 9 *If $\Omega \subset \mathbb{R}^n$ is open then there is a countable collection $\{Q_j\}$ of dyadic cubes with disjoint interiors such that*

$$1 \leq \frac{\operatorname{dist}(Q_j, \partial\Omega)}{\sqrt{n}\, l(Q_j)} \leq 4 \tag{4}$$

and if $Q_j \bigcap Q_k \neq \emptyset$

$$\frac{1}{4} \leq \frac{l(Q_j)}{l(Q_k)} \leq 4. \tag{5}$$

The collection $\mathcal{W} = \{Q_j\}$ is called the Whitney decomposition of Ω.

Notice in particular that if Q is the Whitney cube of Ω containing x then $4\sqrt{n}\, l(Q) \geq \operatorname{dist}(Q, \partial\Omega) \geq \operatorname{dist}(x, \partial\Omega) - \sqrt{n}\, l(Q)$, so that

$$l(Q) \geq \operatorname{dist}(x, \partial\Omega)/(5\sqrt{n}) \tag{6}$$

The geometry of a locally uniform domain Ω may conveniently be described using Whitney cubes. Following Jones [11], we say two Whitney cubes *touch* if their intersection contains a face of one or both cubes, and that a finite sequence $S_1, \ldots, S_m$ of cubes forms a *chain* if S_j and S_{j+1} touch for $j = 1, \ldots, m$. A chain $S = S_1, \ldots, S_m = S'$ is said to *connect* S to S' and have *length* m. We define $\mathcal{W}_1$ to be the collection of Whitney cubes of Ω, and $\mathcal{W}_2$ to be those of the interior of $\mathbb{R}^n \setminus \Omega$.

Connecting two cubes of comparable size

Lemma 10 *Let S and S' be cubes from $\mathcal{W}_1$ that have comparable sizes and separation, that is*

$$\frac{1}{C} \leq \frac{l(S)}{l(S')} \leq C, \quad \frac{1}{C} \leq \frac{|x_S - x_{S'}|}{l(S)} \leq C, \quad \frac{1}{C} \leq \frac{|x_S - x_{S'}|}{l(S')} \leq C$$

where x_S and $x_{S'}$ are the centers of S and S' respectively. Suppose also that $l(S), l(S')$ and $|x_S - x_{S'}|$ are all less than δ. Then there are constants $C_1 = C_1(n, C, \epsilon)$ and $C_2 = C_2(n, C)$ and a connecting chain $S = S_1, \ldots, S_m = S'$ of cubes from $\mathcal{W}_1$ with length $m \leq C_1$, and such that every cube S_j in the chain satisfies

$$\frac{\epsilon}{C_2} \leq \frac{l(S_j)}{l(S)} \leq \frac{C_2}{\epsilon} \quad \textit{and} \quad \frac{\epsilon}{C_2} \leq \frac{l(S_j)}{l(S')} \leq \frac{C_2}{\epsilon} \tag{7}$$

PROOF. This lemma is a variant of Lemma 2.4 from [11], and has the same proof. Since $|x_S - x_{S'}| < \delta$, there is a rectifiable curve γ joining x_S to $x_{S'}$ with property (1). Let z be a point on γ. If $z \in S$ (or S') then $\mathrm{dist}(z, \partial\Omega) \geq l(S)/2$ (respectively $l(S'/2)$). If not, then $|z - x_S| > l(S)/2$ and $|z - x_{S'}| > l(S')/2$, so by (1)

$$\mathrm{dist}(z, \partial\Omega) \geq \frac{\epsilon l(S) l(S')}{4|x_S - x_{S'}|} \geq C_3 \epsilon l(S)$$

Conversely $\mathrm{dist}(z, \partial\Omega) \leq \mathrm{dist}(x_S, \partial\Omega) + l(\gamma) \leq 4\sqrt{n} l(S) + |x_S - x_{S'}|/\epsilon$. Using (4) and (6) we see that any $S_j \in \mathcal{W}_1$ which meets γ satisfies (7). From the collection of cubes meeting γ we then extract a finite chain joining S to S'; the bounds (7) and the length bound on γ ensure that this chain has length at most some $C_1(n, C, \epsilon)$.

Connecting a small cube to a large cube

In this context a *large* cube is one having length comparable to $\epsilon\delta/\sqrt{n}$. This is the largest size of cube which may be found all along the boundary, in the sense that any cube from Ω (or even any point of $\partial\Omega$) may be connected to a cube of this size by an arc of comparable length, and thence by a chain with known structure. This is made precise in the following lemmas, and illustrated in Figure 1.

Lemma 11 *Let $x \in \Omega$ satisfy $\mathrm{dist}(x, \partial\Omega) < \epsilon\delta/(20\sqrt{n})$. Then there is $S \in \mathcal{W}_1$ with $l(S) \geq \epsilon\delta/(20\sqrt{n})$, such that x may be connected to the center x_S of S by a rectifiable curve lying within distance $\epsilon\delta$ of $\partial\Omega$ and of length at most δ/ϵ.*

PROOF. If x already lies in a Whitney cube S of side length at least $\epsilon\delta/(20\sqrt{n})$ then we need only connect x to the center x_S by a straight line. It cannot lie in a larger cube as it is too close to $\partial\Omega$. Hence we assume that the Whitney cube containing x has length less than $\epsilon\delta/(20\sqrt{n})$.

Since Ω is connected and of diameter at least 1 there is a point $y \in \Omega$ such that $|x - y| = \delta$. From Definition 5 there is a rectifiable curve γ of length at most δ/ϵ joining x to y, and in particular containing a point z equidistant from both x and y. It is immediate that $|z - x| = |z - y| \geq \delta/2$, so at z we have by (1)

$$\mathrm{dist}(z, \partial\Omega) \geq \frac{\epsilon |z - x||z - y|}{|x - y|} \geq \frac{\epsilon\delta}{4}$$

and therefore by (6) that $S' \ni z$ has length $l(S') \geq \epsilon\delta/20\sqrt{n}$.

It is now legitimate to take the first cube of length $\epsilon\delta/(20\sqrt{n})$ encountered as we traverse γ from x to y . Call this cube S. The piece of γ connecting x to S lies entirely within cubes smaller than $\epsilon\delta/(20\sqrt{n})$, hence within distance $\epsilon\delta$ of the boundary. The cube S has $l(S) \geq \epsilon\delta/(20\sqrt{n})$ but must be adjacent to a cube with length smaller than that, so by (4) and (5) we have $l(S) < \epsilon\delta/(5\sqrt{n})$ and it is also within distance $\epsilon\delta$ of the boundary. Moreover the curve from x to S is no longer than that from x to z, so has length at most $\delta/\epsilon - \delta/2$. We can adjoin to this curve a line segment from its endpoint on ∂S to the center x_S and have thereby connected x to x_S by a curve of total length at most $\delta/\epsilon - \delta/2 + \epsilon\delta/5 \le \delta/\epsilon$.

Lemma 12 *Let $Q \in \mathcal{W}_2$ with $l(Q) \le \epsilon\delta/(200n)$. Then there is a Whitney cube $S^* \in \mathcal{W}_1$ with*

$$2\sqrt{n} \le \frac{l(S^*)}{l(Q)} \le 8\sqrt{n} \tag{8}$$

$$\mathrm{dist}(Q, S^*) \le \frac{Cn}{\epsilon} l(Q) \tag{9}$$

and a chain $\{S^* = S_1, S_2, \ldots, S_m = S\}$ with $l(S) \ge \epsilon\delta/(20\sqrt{n})$ and having the property that

$$\frac{\epsilon}{Cn} \le \frac{l(S_j)}{\mathrm{dist}(Q, S_j)} \le 1 \tag{10}$$

where C is a constant independent of n and ϵ.

PROOF. Using the basic properties of the Whitney decomposition we choose a point $x \in \Omega$ such that $\mathrm{dist}(x, x_Q) \le 5\sqrt{n}l(Q)$ and $\mathrm{dist}(x, \partial\Omega) < l(Q)$. From this point we apply Lemma 11 and obtain a curve γ connecting x to a point x_S which is the center of a Whitney cube S with $l(S) \ge \epsilon\delta/(20\sqrt{n})$.

Consider the collection of cubes from $\mathcal{W}_1$ that intersect γ. This collection contains a chain of cubes from x to S, so we need only see that there is an appropriate starting cube on this chain and that the estimates hold. Observe that the chain contains a cube of length at most $\mathrm{dist}(x, \partial\Omega) < l(Q)$ and also a cube of length $l(S) > 8\sqrt{n}l(Q)$, hence by property (5) of the Whitney decomposition it certainly contains one cube of length between $2\sqrt{n}l(Q)$ and $8\sqrt{n}l(Q)$. Ordering the cubes along the chain beginning at x we call the last cube of this length S^*. Since $S^* \ne S$ we can apply (6) and (1) to $z \in \gamma \cap S^*$ to obtain

$$40nl(Q) \ge 5\sqrt{n}l(S^*) \ge \mathrm{dist}(z, \partial\Omega) \ge \frac{\epsilon|z-x||z-x_S|}{|x-x_S|} \ge \frac{\epsilon|z-x|}{2}$$

so that $|z - x| \le 80nl(Q)/\epsilon$ and therefore $\mathrm{dist}(Q, S^*) \le Cnl(Q)/\epsilon$

Let $\{S_j\}$ be the chain from S^* to S. For any $z \in \gamma \cap S_j$

$$5\sqrt{n}l(S_j) \ge \mathrm{dist}(S_j, \partial\Omega) + \sqrt{n}l(S_j) \ge \mathrm{dist}(z, \partial\Omega)$$

therefore applying the estimate (1) in the case $S_j \ne S$

$$5\sqrt{n}l(S_j) \ge \mathrm{dist}(z, \partial\Omega) \ge \frac{\epsilon|z-x||z-x_S|}{|x-x_S|} \ge \frac{\epsilon}{2}|z-x| \ge \frac{\epsilon}{2}(|z-x_Q| - |x_Q - x|)$$

whereupon

$$\frac{10\sqrt{n}}{\epsilon} l(S_j) \geq \frac{\epsilon}{2}(\mathrm{dist}(x_Q, S_j) - 5\sqrt{n}l(Q)) \geq \mathrm{dist}(Q, S_j) - 6\sqrt{n}l(Q)$$

and using the fact that $l(S_j) \geq l(S^*) \geq 2\sqrt{n}l(Q)$ we have

$$\mathrm{dist}(Q, S_j) \leq \frac{10\sqrt{n}}{\epsilon} l(S_j) + 12nl(Q) \leq \frac{Cn}{\epsilon} l(S_j)$$

from which (10) follows for all cubes but S. For the cube S we can repeat the above computation for $z \in \partial S$ rather than $z \notin S$. All of the estimates are identical.

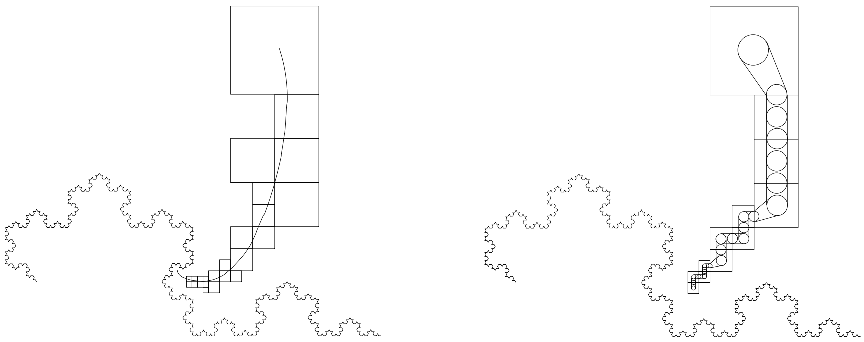

Fig. 1. Construction of a chain of cubes and the twisting cone Γ.

Tubes and Twisting Cones

In order to simplify some of our proofs we perform an elementary construction that gives a region inside the chains constructed above and on which it is easy to propagate the estimates we shall need later.

Let $\{S_j\}$ be a chain of Whitney cubes with no repeated cubes. Let a_j be the center of the cube S_j and b_j be the center of the face $S_j \cap S_{j+1}$. We trace out a piecewise linear curve γ through these points in the order $a_1, b_1, a_2, \ldots, b_{m-1}, a_m$. At each point $x \in \gamma$ define a radius $s(x)$ which varies linearly between points a_j and b_j and is such that $s(a_j) = \frac{1}{2} l(S_j)$ and $s(b_j) = \frac{1}{2}\min\{l(S_j), l(S_{j+1})\}$. Finally, let Γ be the set of points that lie within radius $s(x)$ of some $x \in \gamma$. The result is shown in Figure 1.

Lemma 13 *If $y \in \Gamma \cap S_j$ then $B\left(y, \sqrt{n}l(Q)\right) \subset S_{j-1} \cup S_j \cup S_{j+1}$.*

PROOF. All points x with $|x - y| \leq \frac{1}{2}\min\{l(S_{j-1}), l(S_j), l(S_{j+1})\}$ are in $S_{j-1} \cup S_j \cup S_{j+1}$. However in the proof of Lemma 12 the smallest of the cubes S_j was S^* and had length at least $2\sqrt{n}l(Q)$ by (8).

If our chain is one of those described in Lemma 10 than the set Γ has radius comparable to the lengths of the cubes at its ends, with bounds depending only on ϵ, n, and the constant C in the lemma. Such Γ are called *tubes*.

In the case that the chain connects a small cube to a large cube, as in Lemma 12, we have instead that Γ is a *twisting cone*. The name describes the fact that the radius $s(x)$ is comparable to the function that grows linearly along γ and is equal to $l(S_1)$ at one end and $l(S_m)$ at the other.

Estimation along Tubes and Twisting Cones

Part of our reason for introducing tubes and twisting cones was that these are the type of sets on which we may iterate the classical Poincaré inequality to estimate the behavior of a function in terms of its weak derivatives. We state the usual Poincaré inequality on a ball as a theorem; it is proven in most standard references, for example it appears as Theorem 6.30 in [1], and as Lemma 1.1.11 in [14].

Theorem 14 *If* $f \in L^p_k(B(0,r))$ *satisfies*

$$\int_{B(0,r)} D^\alpha f = 0 \quad \textit{for all } |\alpha| \leq k-1 \tag{11}$$

then for all $1 \leq p \leq \infty$

$$||f||_{L^p(B(0,r))} \leq C(k) r^k ||\nabla^k||_{L^p(B(0,r))} \tag{12}$$

We note in particular that from any $f \in L^p_k$ we may subtract the polynomial

$$P(x) = \sum_{|\alpha|\leq k-1} \frac{x^\alpha}{\alpha!} \fint_B D^\alpha(\xi)\, d\xi \tag{13}$$

and thereby ensure $f(x) - P(x)$ satisfies (11). We call $P(x)$ the polynomial *fitted* to f on B.

Before giving our estimate for the behavior of f along a twisting cone Γ we fix some notation. Recall that Γ is centered on a piecewise linear curve γ and contained in a chain of cubes $\{S_j\}$. The ordered vertices of γ, called a_j and b_j in the definition of a twisting cone, will here be denoted $\{z_j\}$. There is a radius $s(z)$ at each $z \in \gamma$ comparable to the size of the enclosing cube $S_j \ni z$. We use $B_j = B(z_j, s(z_j))$ for the balls around the vertices and $P_k(B_j; f)$ for the polynomial of degree k fitted to f on B_j.

Lemma 15 *Let* $\{S_j\}$ *be a chain of Whitney cubes as in Lemma 10 or Lemma 12, and* Γ *be the tube or twisting cone around* γ *that is contained in the chain. Let* $s(z)$ *be* *the radius of Γ at $z \in \gamma$, write z_0 and z_m for the endpoints of γ, and $B_0 = B(z_0, s(z_0))$ and $B_m = B(z_m, s(z_m))$ for the balls around these endpoints.*

Consider $f \in L^p_k(\Omega)$. If $P(x)$ is the polynomial of degree $k-1$ fitted to f on the ball B_0 then there are constants $C = C(n, \epsilon, k, p)$ such that if $1 \le p < \infty$

$$\left\| f(x) - P(x) \right\|_{L^p(B_m)} \le C\,(l(S_m))^{k-1} \sum_{j=1}^{m} l(S_j) \left(\frac{l(S_m)}{l(S_j)} \right)^{n/p} \left\| \nabla^k f(y) \right\|_{L^p(S_j)} \tag{14}$$

while for $p = \infty$

$$\left\| f(x) - P_Q(x) \right\|_{L^\infty(B_m)} \le C\, l(S_m)^k \left\| \nabla^k f \right\|_{L^\infty(\Omega)} \tag{15}$$

PROOF. Suppose $1 \le p < \infty$. We begin by examining a special case that occurs along each segment of the curve γ. Let $k = 1$ and consider the set consisting of the convex hull of the unit ball B centered at the origin and a ball of radius $(1+\lambda)$ centered at the point a. Use $s(t) = 1 + \lambda t$ for the radius at position ta along the central axis. This is a convex set, so smooth functions are dense in the Sobolev functions (by an easy mollification argument) and it suffices to prove our estimates under the assumption that f is differentiable. For each $\xi \in B(0,1)$ we have

$$f(a + (1+\lambda)\xi) - f(\xi) = \int_0^1 \frac{\partial f}{\partial t}(\xi + (a + \lambda\xi)t)\, dt = \int_0^1 \nabla f(\xi + (a+\lambda\xi)t) \cdot (a + \lambda\xi)\, dt$$

from which by Jensen's inequality and the fact $|\xi| \le 1$

$$\begin{aligned} \int_B \left| f(a + (1+\lambda)\xi) - f(\xi) \right|^p d\xi &\le \int_B \int_0^1 \left| \nabla f((1+\lambda t)\xi + at) \right|^p |a + \lambda\xi|^p \, dt\, d\xi \\ &\le (|a| + \lambda)^p \int_0^1 \int_{B(at,1)} \left| \nabla f(s(t)\xi) \right|^p d\xi\, dt \\ &\le (|a| + \lambda)^p \int_0^1 \int_{B(at,s(t))} |\nabla f(y)|^p \frac{dy}{(s(t))^n}\, dt \end{aligned} \tag{16}$$

However the usual Poincaré theorem for $k = 1$ states

$$\int_{B(0,1)} \left| f(\xi) - \fint_{B(0,1)} f(x)\, dx \right|^p d\xi \le C \int_{B(0,1)} |\nabla f(\xi)|^p \, d\xi \tag{17}$$

And since the average of f is precisely the zero order polynomial approximation $P_0(B; f)$, we may combine this with (16), (17) and a change of variables to obtain

$$\begin{aligned}
\left(\fint_{B(a,1+\lambda)} \left| f(y) - P_0(B; f)\right|^p dy\right)^{1/p} \\
&= \left(\int_B \left| f(a + (1+\lambda)\xi) - P_0(B; f)\right|^p d\xi\right)^{1/p} \\
&\leq C\|\nabla f\|_{L^p(B)} + \left(\int_B \left| f(a + (1+\lambda)\xi) - f(\xi)\right|^p d\xi\right)^{1/p} \\
&\leq C\|\nabla f\|_{L^p(B)} + (|a| + \lambda)\left(\int_0^1 \int_{B(at,s(t))} |\nabla f(y)|^p \frac{dy}{(s(t))^n}\, dt\right)^{1/p} \qquad (18)
\end{aligned}$$

If we apply the Poincaré estimate (17) again, but this time on the ball $B' = B(a, 1 + \lambda)$ we have

$$\fint_{B'} \left| f(y) - P_0(B'; f)\right|^p dy = \fint_{B'} \left| f(y) - \fint_{B'} f(x)\, dx\right|^p dy \leq C(1+\lambda)^p \fint_{B'} |\nabla f(x)|^p\, dx$$

and in conjunction with (18) we have shown

$$\begin{aligned}
\left|P_0(B'; f) - P_0(B; f)\right| &\leq C(1+\lambda)\left(\fint_{B'} |\nabla f(y)|^p\, dy\right)^{1/p} + C\left(\fint_B |\nabla f(y)|\, dy\right)^{1/p} \\
&\quad + (|a| + \lambda)\left(\int_0^1 \fint_{B(at,s(t))} |\nabla f(y)|^p\, dy\, dt\right)^{1/p} \qquad (19)
\end{aligned}$$

We think of Γ as decomposed into a union of sets having the geometry just discussed, so $\Gamma = \cup\Gamma_l$ where Γ_l is the convex hull of $B(z_l, s(z_l))$ and $B(z_{l+1}, s(z_{l+1}))$. The estimate (19) applies to each Γ_l in the form

$$\begin{aligned}
\left|P_0(B_l; f) - P_0(B_{l-1}; f)\right| &\leq Cs(z_l)\left(\fint_{B_l} |\nabla f(y)|^p\, dy\right)^{1/p} + Cs(z_{l-1})\left(\fint_{B_{l-1}} |\nabla f(y)|\, dy\right)^{1/p} \\
&\quad + |z_l - z_{l-1}|\left(\int_{z_{l-1}}^{z_l} \fint_{B(z,s(z))} |\nabla f(y)|^p\, dy\, \frac{|dz|}{|z_l - z_{l-1}|}\right)^{1/p} \\
&\leq Cs(z_l)\left(\fint_{B_l} |\nabla f(y)|^p\, dy\right)^{1/p} + Cs(z_{l-1})\left(\fint_{B_{l-1}} |\nabla f(y)|\, dy\right)^{1/p} \\
&\quad + C|z_l - z_{l-1}|\left(\fint_{\Gamma_{l-1}} |\nabla f(y)|^p\, dy\right)^{1/p} \qquad (20)
\end{aligned}$$

and we can write

$$
\begin{aligned}
&\left(\fint_{B_j}\left|f(y)-P_0(B_0;f)\right|^p\,dy\right)^{1/p}\\
&\quad=\left(\fint_{B_j}\left|f(y)-P_0(B_j;f)+\sum_{l=1}^{j}\left(P_0(B_l;f)-P_0(B_{l-1};f)\right)\right|^p dy\right)^{1/p}\\
&\quad\le\left(\fint_{B_j}\left|f(y)-P_0(B_j;f)\right|^p dy\right)^{1/p}+\sum_{l=1}^{j}\left|P_0(B_l;f)-P_0(B_{l-1};f)\right|\\
&\quad\le C\sum_{l=1}^{j}s(z_l)\left(\fint_{B_l}|\nabla f(y)|^p\,dy\right)^{1/p}+C\sum_{l=1}^{j}|z_l-z_{l-1}|\left(\fint_{\Gamma_{l-1}}|\nabla f(y)|^p\,dy\right)^{1/p}\\
&\quad\le C\sum_{l=1}^{j}|z_l-z_{l-1}|\left(\fint_{\Gamma_{l-1}}|\nabla f(y)|^p\,dy\right)^{1/p}
\end{aligned}
\tag{21}
$$

where the last step uses the fact that

$$
\begin{aligned}
s(z_l)^p\fint_{B_l}|\nabla f(y)|^p\,dy&=\left(\frac{s(z_l)}{|z_l-z_{l-1}|}\right)^p\frac{|\Gamma_{l-1}|}{|B_l|}|z_l-z_{l-1}|^p\fint_{\Gamma_{l-1}}|\nabla f(y)|^p\,dy\\
&\le C(p)|z_l-z_{l-1}|^p\fint_{\Gamma_{l-1}}|\nabla f(y)|^p\,dy
\end{aligned}
$$

This concludes our discussion of the case $k=1$.

Fortunately the case of general k is not dissimilar from what we have done for $k=1$. Let γ_j be the arc of γ up to z_j and suppose inductively that for any smooth function g and any ball $B=B(x,s(x))$ along the segment $[z_{j-1},z_j]$ we have

$$
\begin{aligned}
&\left(\fint_{B}\left|g(y)-P_{k-2}(B_0;f)\right|^p dy\right)^{1/p}\\
&\qquad\le C\left(l(\gamma_j)\right)^{k-2}\sum_{l=1}^{j}|z_l-z_{l-1}|\left(\fint_{\Gamma_{l-1}}\left|\nabla^{k-1}g(y)\right|^p dy\right)^{1/p}.
\end{aligned}
\tag{22}
$$

Note also from (13) that the components of $P_{k-2}(B;\nabla f)$ coincide with those of $\nabla P_{k-1}(B;f)$.

Returning to the case of a conical piece of Γ with notation as before, we follow the same method as in (16) but for the function $f-P_{k-1}(B;f)$ and using our observation about $\nabla P_{k-1}(B;f)$. Here $a=z_j-z_{j-1}$ and $1+\lambda=s(z_j)/s(z_{j-1})$, so that we are moving on the cone from B_{j-1} to B_j.

$$
\begin{aligned}
&\fint_{B_{j-1}} \left|(f - P_{k-1}(B; f))\,(a + (1+\lambda)\xi) - (f - P_{k-1}(B; f))\,(\xi)\right|^p \, d\xi \\
&\quad \le (|a| + \lambda)^p \int_0^1 \fint_{B_{j-1}} \left|\nabla\,(f - P_{k-1}(B; f))\,((1+\lambda t)\xi + at)\right|^p \, d\xi \, dt \\
&\quad = (|a| + \lambda)^p \int_0^1 \fint_{B_{j-1}} \left|(\nabla f - P_{k-2}(B; \nabla f))\,((1+\lambda t)\xi + at)\right|^p \, d\xi \, dt \\
&\quad \le C|z_j - z_{j-1}|^p \int_0^1 \fint_{B(at, 1+\lambda t)} \left|(\nabla f - P_{k-2}(B; \nabla f))\,(y)\right|^p \, dy \, dt
\end{aligned}
$$

whence by our inductive assumption applied to $g = \nabla f$, and using that $at \in [z_{j-1}, z_j]$

$$
\begin{aligned}
&\le C|z_j - z_{j-1}|^p l(\gamma_j)^{(k-2)p} \int_0^1 \left[\sum_{l=1}^{j} |z_l - z_{l-1}| \left(\fint_{\Gamma_{l-1}} |\nabla^{k-1} g(y)|^p \, dy \right)^{1/p} \right]^p dt \\
&\le C l(\gamma_j)^{(k-2)p} |z_j - z_{j-1}|^p \left[\sum_{l=1}^{j} |z_l - z_{l-1}| \left(\fint_{\Gamma_{l-1}} |\nabla^{k} f(y)|^p \, dy \right)^{1/p} \right]^p
\end{aligned}
$$

since the integrand is no longer dependent on t. We use this to write

$$
\begin{aligned}
&\left(\fint_{B_j} \left|(f - P_{k-1}(B; f))\,(y)\right|^p \, dy \right)^{1/p} \\
&\qquad = \left(\fint_{B_{j-1}} \left|(f - P_{k-1}(B; f))\,(a + \lambda\xi)\right|^p \, d\xi \right)^{1/p} \\
&\qquad \le \left(\fint_{B_{j-1}} |f(\xi) - P_{k-1}(B; f)(\xi)|^p \, d\xi \right)^{1/p} \\
&\qquad\quad + C\, l(\gamma_j)^{(k-2)} |z_j - z_{j-1}| \sum_{l=1}^{j} |z_l - z_{l-1}| \left(\fint_{\Gamma_{l-1}} |\nabla^k f(y)|^p \, dy \right)^{1/p} \qquad (23)
\end{aligned}
$$

It is clear from inductive application of (23) and a single use of the Poincaré inequality that

$$
\begin{aligned}
&\left(\fint_{B_m} \left| (f - P_{k-1}(B; f))\,(y) \right|^p dy \right)^{1/p} \\
&\quad \le \left(\fint_{B_1} |f(\xi) - P_{k-1}(B; f)(\xi)|^p \, d\xi \right)^{1/p} \\
&\qquad + C \sum_{j=1}^{m} l(\gamma_j)^{(k-2)} |z_j - z_{j-1}| \sum_{l=1}^{j} |z_l - z_{l-1}| \left(\fint_{\Gamma_{l-1}} |\nabla^k f(y)|^p \, dy \right)^{1/p} \\
&\quad \le C(s(z_1))^k \left(\fint_{B} |\nabla^k f(y)|^p \, dy \right)^{1/p} \\
&\qquad + C \left(\sum_{j=1}^{m} l(\gamma_j)^{(k-2)} |z_j - z_{j-1}| \right) \sum_{l=1}^{m} |z_l - z_{l-1}| \left(\fint_{\Gamma_{l-1}} |\nabla^k f(y)|^p \, dy \right)^{1/p} \\
&\quad \le C(l(\gamma_m))^{(k-1)} \sum_{l=1}^{m} |z_l - z_{l-1}| \left(\fint_{\Gamma_{l-1}} |\nabla^k f(y)|^p \, dy \right)^{1/p}
\end{aligned}
\tag{24}
$$

Comparing this to (22) and using the base case $k = 1$ established in (21) we see that (24) is true for all k.

It is not difficult to pass from (24) to the desired estimate (14). The sets Γ_l are contained in cubes of the chain $\{S_j\}$. If $\Gamma_l \cap S_j \neq \emptyset$ then $|\Gamma_l|$ and $|S_j|$ are comparable and the length $|z_l - z_{l-1}|$ is is comparable to $l(S_j)$. Moreover the length $l(\gamma_j)$ is comparable to $l(S_j)$ with a constant depending on ϵ, because the length of a subarc of γ is comparable to the separation of the endpoints and we know (10). Multiplying both sides of (24) by $|B_m|^{1/p}$ and rewriting the bound in terms of $l(S_j)$ we have

$$
\left\| f - P_{k-1}(B; f) \right\|_{L^p(B_m)} \le C \left(l(S_m) \right)^{k-1} \sum_{j=1}^{m} l(S_j) \left(\frac{l(S_m)}{l(S_j)} \right)^{n/p} \left\| \nabla^k f(y) \right\|_{L^p(S_j)}
$$

This concludes the proof for the case $1 \le p < \infty$.

When $p = \infty$ the argument is considerably simpler. It is a well known consequence of the Sobolev Embedding Theorem that $f \in L^\infty_k(\Omega)$ has a representative for which $\nabla^{k-1} f$ is Lipschitz on balls contained in Ω, with Lipschitz norm $\|\nabla^k f\|_{L^\infty(\Omega)}$. Integrating $\nabla^k f$ along a rectifiable curve then gives bounds for lower order derivatives as is usual in Taylor's Theorem. As the uniform domain condition ensures that any x and y with $|x - y| < \delta$ are joined by a large number of rectifiable curves of length not exceeding $C(\epsilon)|x - y|$, we conclude immediately that

$$
\left| (f(x) - P_Q(x)) - (f(y) - P_Q(y)) \right| \le C(\epsilon, k) |x - y|^k \|\nabla^k f\|_{L^\infty(\Omega)}
$$

This implies both that $\left| f(x) - P_Q(x) \right|$ is bounded by $C\|\nabla^k f\|\, l(S_0)^k$ on B_0 and that $\left| f(x) - f(y) \right|$ is bounded by $C\|\nabla^k f\|\, l(S_m)^k$ for $x \in B_0$ and $y \in B_m$, so (15) follows and the lemma is proven.

Counting Cubes in Tubes and Twisting Cones

In the sequel we shall need to perform estimates along twisting cones and families of tubes for each Whitney cube from $\mathcal{W}_2$. This will cause most cubes from $\mathcal{W}_1$ to be counted many times, so we record some bounds on how frequently a single cube occurs.

For the estimate on families of tubes we fix $Q \in \mathcal{W}_2$ and constants C_1 and C_2. Let

$$\mathcal{F}(Q) = \{S_j \in \mathcal{W}_1 : l(S) \geq C_1 l(Q) \text{ and } \operatorname{dist}(S, Q) \leq C_2 l(Q)\}. \tag{25}$$

Any two cubes S_j, S_k from $\mathcal{F}(Q)$ satisfy the conditions of Lemma 10 so they are connected by a chain $\{T_l(S_j, S_k)\}$ containing at most C_3 cubes. There are finitely many cubes in $\mathcal{F}(Q)$, hence

$$\left\| \sum_{S_j, S_k \in \mathcal{F}(Q)} \sum_l \Psi_{T_l(S_j, S_k)}(x) \right\|_{L^\infty} \leq C_4(\epsilon, n, C_1, C_2)$$

where $\Psi_A(x)$ is the characteristic function of the set A. Furthermore the cubes $T_l(S_j, S_k)$ all have length comparable to $l(Q)$ and satisfy $\operatorname{dist}(Q, T_l) \leq C_5 l(Q)$, so chains arising from the above construction applied to the set $\mathcal{F}(Q')$ can only intersect those corresponding to $\mathcal{F}(Q)$ for finitely many choices of Q', and therefore

$$\left\| \sum_{Q \in \mathcal{W}_2} \sum_{S_j, S_k \in \mathcal{F}(Q)} \sum_l \Psi_{T_l(S_j, S_k)}(x) \right\|_{L^\infty} \leq C_6(\epsilon, n, C_1, C_2). \tag{26}$$

For twisting cones the situation is different. A cube $S \in \mathcal{W}_1$ intersects infinitely many twisting cones but only finitely many of any given scale.

Suppose that for each sufficiently small $Q \in \mathcal{W}_2$ we have a corresponding twisting cone Γ_Q. Fix $S \in \mathcal{W}_1$ and let $\mathcal{G}(S)$ be the set of all $Q \in \mathcal{W}_2$ such that $\Gamma_Q \cap S \neq \emptyset$. Since the smallest cube in the chain containing Γ_Q is bounded as in (8) we see that all $Q \in \mathcal{G}(S)$ have $l(Q) \leq C(n, \epsilon) l(S)$. By (10) any such Q has $\operatorname{dist}(Q, S) \leq C l(S)$, and within this distance there are at most $(C2^j)^n$ cubes Q with $l(Q) = 2^{-j} l(S)$, so we have shown

$$\#\{Q \in \mathcal{G}(S) : l(Q) = 2^{-j} l(S)\} \leq C(n, \epsilon) 2^{nj}. \tag{27}$$

3 A Function with Vanishing Moments

We prove that sets similar to twisting cones support smooth, exponentially decaying functions with vanishing moments of all orders. This is the crucial step in defining a degree independent operator, because the convolution of $f \in L^p_k$ with such a function captures information about all orders of polynomial approximation to f.

Theorem 16 *Let $R_0 > 0$ and $\eta < 1$ be fixed constants. Suppose $\Gamma \subset \mathbb{R}^n$ has the property that for every $r \geq R_0$ there is x with $|x| = r$ and $B(x, \eta|x|) \subset \Gamma$. Then there is a smooth function $K(x)$ supported on Γ, and constants C and T depending only on n, η and R_0, such that*

$$\int_{\mathbb{R}^n} x^\alpha K(x)\,dx = \begin{cases} 1 & \text{if} \quad \alpha = (0,\ldots,0) \\ 0 & \text{if} \quad \alpha \in \mathbb{N}^n \setminus \{(0,\ldots,0)\} \end{cases} \tag{28}$$

$$\left|K(x)\right| \leq \kappa(|x|)|x|^{1-n} \tag{29}$$

where

$$\kappa(t) = \exp\left[-\left(\frac{1}{2}\log\frac{t}{T}\right)^{1/2}\exp\left(\frac{1}{2}\log\frac{t}{T}\right)^{1/2}\right]. \tag{30}$$

Theorem 16 is a consequence of the following technical lemma, which describes the desired geometry in more detail.

Lemma 17 *For fixed constants R and j_0, let $r_j = R\exp\left[2\log^2(j + j_0)\right]$. Fix also a constant λ, and suppose that $\Gamma \subset \mathbb{R}^n$ has the property that for each j there is a point $\xi_j \in S^{n-1}$ and $\Lambda_j = S^{n-1} \cap B(\xi_j, \lambda)$ with*

$$\left\{x : r_j \leq |x| \leq r_{j+1} \text{ and } \frac{x}{|x|} \in \Lambda_j\right\} \subset \left(\Gamma \cap \{x : r_j \leq |x| \leq r_{j+1}\}\right).$$

Then there is a smooth function $K(x)$ supported on Γ which has the property (28) *and satisfies the estimate* (29) *with constants C and T depending on n, R, λ and j_0.*

PROOF. We prove that Lemma 17 implies Theorem 16. The assumptions of the theorem readily imply the existence of a constant c with the property that at any radius $r \geq R_0$ there is x with $|x| = r$ and

$$\left\{y : \left(1 - \frac{\eta}{2}\right)r \leq |y| \leq \left(1 + \frac{\eta}{2}\right)r,\ \frac{y}{|y|} \in B\left(\frac{x}{|x|}, c\eta\right)\right\} \subset B(x, \eta|x|) \subset \Gamma$$

From this it suffices that we can choose j_0 such that $r_{j+1}/r_j \le (2+\eta)/(2-\eta)$ for all j, and R such that $r_0 \ge R_0$. The former is equivalent to requiring

$$\exp\left[2\log^2(j+j_0+1) - 2\log^2(j+j_0)\right] \le \frac{2+\eta}{2-\eta}$$

and since $(\log^2(x+1) - \log^2 x)$ is decreasing for $x > 1$ and has limit zero as $x \to \infty$ this may be achieved by taking j_0 sufficiently large. With $R = R_0 \exp[-2\log^2 j_0]$ the latter condition is also satisfied.

The remainder of this section is spent proving Lemma 17. Considering the variation in the radial co-ordinate $|x|$ leads us to examine the existence of smooth functions with vanishing moments on the half-line. This is a classical problem in the theory of moments that was first solved by Stieltjes [19,20]. An elegant proof using complex analysis is in Chapter VI, Section 3.2 of [18]. Unfortunately neither of these arguments adapts well to twisting cones, so we begin with a different approach that allows us greater control over the regions on which individual moments cancel. We then turn to the angular dependence, and the construction of certain functions on the sets Λ_j. These are combined with the functions from the one dimensional case to produce $K(x)$.

Vanishing Moments on the Half-Line

Let $\{r_j\}_{j=0}^{\infty}$ be an increasing sequence of positive real numbers. We partition $I = [r_0, \infty)$ into the intervals $I_j = [r_j, r_{j+1})$. Our first goal is to construct smooth functions ψ_j which have a finite number of vanishing moments and which are supported on the intervals I_j. From the functions ψ_j we will then inductively construct a function Ψ satisfying (28). This will require knowing estimates for the higher order moments of the ψ_j.

Some Building Blocks

Consider for each $j \in \mathbb{N}$, $j \ne 0$ the function

$$\chi_j(s) = \begin{cases} C_j \exp\left(\dfrac{j}{s^2-1}\right) & s \in (-1,1) \\ 0 & \text{otherwise} \end{cases}$$

where C_j is chosen so that $\int \chi_j = 1$. For $j = 0$ set $\psi_0 = \psi_1$. These functions are C^∞ on the real line and are supported on $[-1,1]$. It is elementary to show $C_j \le e^{4j/3}$.

We use ϕ_j to denote the function obtained by translating and scaling χ_j to the interval I_j such that ϕ_j is C^∞, supported on I_j, and has $\int \phi_j = 1$.

$$\phi_j(r) = \frac{2}{(r_{j+1} - r_j)} \chi_j \left(\frac{2r}{r_{j+1} - r_j} - \frac{r_{j+1} + r_j}{r_{j+1} - r_j} \right) \tag{31}$$

Now we make our main definition for this section. The j-th building block function, supported on the interval I_j, is

$$\psi_j(r) = \frac{(-1)^j}{j!} \left(\frac{\partial}{\partial r} \right)^j \phi_j(r) \tag{32}$$

This definition is related to the classical Rodrigues formula for the Legendre polynomials. As in the theory of orthogonal polynomials, its practical application comes from the ease with which we may calculate the moments $\mu_{j,k}$ of ψ_j using integration by parts. We differentiate r^k and integrate $\psi_j(r)$ as many as j times. Notice that at each stage the boundary terms vanish because they are multiples of derivatives of ϕ_j at the endpoints of I_j, so we obtain

$$\mu_{j,k} = \int_I r^k \psi_j(r)\, dr = \begin{cases} 0 & \text{if } k < j \\ 1 & \text{if } k = j \\ \binom{k}{j} \int_{I_j} r^{k-j} \phi_j(r)\, dr & \text{if } k > j \end{cases} \tag{33}$$

At times we will need the following elementary estimate for the $\mu_{j,k}$ with $k > j$

$$|\mu_{j,k}| \leq \binom{k}{j} r_{j+1}^{k-j} \tag{34}$$

Bounds for the building blocks

As our construction will involve adding and subtracting multiples of the functions ψ_j it will be important that we know how the L^∞ norm of ψ_j depends on j.

Lemma 18 *The functions ψ_j satisfy*

$$|\psi_j(r)| \leq \left(\frac{20}{r_{j+1} - r_j} \right)^{j+1} \tag{35}$$

PROOF. By (31), (32) and the linearity of the change of variables we find that it suffices to know a bound for the j-th derivative of χ_j:

$$\begin{aligned}\psi_j(r) &= \frac{(-1)^j}{j!}\frac{2}{(r_{j+1}-r_j)}\left(\frac{d}{dr}\right)^j \chi_j\left(\frac{2r}{r_{j+1}-r_j} - \frac{r_{j+1}+r_j}{r_{j+1}-r_j}\right) \\ &= \frac{(-1)^j}{j!}\left(\frac{2}{(r_{j+1}-r_j)}\right)^{j+1}\left(\frac{d}{ds}\right)^j \chi_j(s) \end{aligned} \tag{36}$$

Rewriting the definition of $\chi_j(s)$ as

$$\chi_j(s) = C_j \exp\left(\frac{j}{s^2-1}\right) = C_j \exp\left(\frac{j}{2(s-1)}\right)\exp\left(\frac{-j}{2(s+1)}\right) \tag{37}$$

we may proceed by differentiating the product to obtain

$$C_j^{-1}\left(\frac{d}{ds}\right)^j \chi_j(s) = \sum_{k=0}^{j}\binom{j}{k}\cdot\left(\frac{d}{ds}\right)^k \exp\left(\frac{j}{2(s-1)}\right)\cdot\left(\frac{d}{ds}\right)^{j-k}\exp\left(\frac{-j}{2(s+1)}\right)$$

It is elementary but tedious to obtain bounds for these derivatives. The terms that arise when we expand using the Leibnitz rule are products involving $(s-1)^{-l}\exp(j/2(s-1))$. We compute

$$\frac{d}{ds}\left[\frac{1}{(s-1)^l}\exp\left(\frac{j}{2(s-1)}\right)\right] = \frac{-l}{(s-1)^{l+1}}\exp\left(\frac{j}{2(s-1)}\right) + \frac{-j}{2(s-1)^{l+2}}\exp\left(\frac{j}{2(s-1)}\right)$$

Grouping such terms according to the homogeneity l allows us to describe all terms that arise in computing the k-th derivative. There are a total of 2^{k-1} terms and the homogeneity of a term depends on the pattern of differentiations that produced it. If l of these fell on the powers of $(s-1)$ and $(k-l)$ on the exponential factor, then the result has homogeneity $2(k-l)+l = 2k-l$. There are $\binom{k-1}{l}$ terms of this homogeneity and it is easy to deduce that the coefficients of each contain a factor of $(-j/2)^{k-l}$ from differentiation of the exponentials. The coefficients obtained by differentiating the powers are no larger than $(2k)^l$.

Now we estimate the size of a term with fixed homogeneity. As there is a trivial estimate on $[-1,0]$ we look for the maximum on $[0,1)$. Observe that for a positive value of $2k-l$

$$\begin{aligned}\log\left|\frac{1}{(s-1)^{2k-l}}\exp\left(\frac{j}{2(s-1)}\right)\right| &= -(2k-l)\log(1-s) + \frac{j}{2(s-1)} \\ \frac{d}{ds}\log\left|\frac{1}{(s-1)^{2k-l}}\exp\left(\frac{j}{2(s-1)}\right)\right| &= \frac{(2k-l)}{(1-s)} - \frac{j}{2(s-1)^2}\end{aligned}$$

so that this expression has a unique critical point in $[0,1)$ at $j/2(s-1) = -(2k-l)$. It follows that we have the bound

$$\left|\frac{1}{(s-1)^{2k-l}}\exp\left(\frac{j}{2(s-1)}\right)\right| \leq \begin{cases} \left(\dfrac{2(2k-l)}{je}\right)^{2k-l} & \text{if } 2(2k-l) \geq j \\ e^{-j/2} & \text{if } 2(2k-l) < j \end{cases} \tag{38}$$

where these maxima occur at the critical point and at 0 respectively.

For $k < j/4$ we use the second estimate in (38) to obtain

$$\begin{aligned}\left|\left(\frac{d}{ds}\right)^k \exp\left(\frac{j}{2(s-1)}\right)\right| &\leq e^{-j/2}\sum_{l=0}^{k-1}\binom{k-1}{l}(2k)^l\left(\frac{j}{2}\right)^{k-l} \\ &\leq e^{-j/2}\left(2k+\frac{j}{2}\right)^k \leq e^{-j/2}j^k\end{aligned}$$

For $k \geq j/2-1$ we have $2k - j/2 \geq k-1 \geq l$ and therefore the first estimate in (38) is used.

$$\begin{aligned}\left|\left(\frac{d}{ds}\right)^k \exp\left(\frac{j}{2(s-1)}\right)\right| &\leq \sum_{l=0}^{k-1}\binom{k-1}{l}(2k)^l\left(\frac{j}{2}\right)^{k-l}\left(\frac{2(2k-l)}{je}\right)^{2k-l} \\ &\leq \left(\frac{4k}{je}\right)^k \sum_{l=0}^{k-1}\binom{k-1}{l}(2k)^l\left(\frac{j}{2}\right)^{k-l}\left(\frac{4k}{je}\right)^{k-l} \\ &= \left(\frac{4k}{je}\right)^k \sum_{l=0}^{k-1}\binom{k-1}{l}(2k)^l\left(\frac{2k}{e}\right)^{k-l} \\ &\leq \left(\frac{4k}{je}\right)^k\left(\frac{e+1}{e}\right)^k(2k)^k \leq C^k\left(\frac{k^2}{j}\right)^k\end{aligned}$$

Finally if $j/4 \leq k < j/2-1$ we use both of the above

$$\begin{aligned}\left|\left(\frac{d}{ds}\right)^k \exp\left(\frac{j}{2(s-1)}\right)\right| &\leq \left(\frac{4k}{je}\right)^k \sum_{l=0}^{2k-j/2}\binom{k-1}{l}(2k)^l\left(\frac{2k}{e}\right)^{k-l} \\ &\quad + e^{-j/2}\sum_{l=2k-j/2}^{k-1}\binom{k-1}{l}(2k)^l\left(\frac{j}{2}\right)^{k-l} \\ &\leq C^k\left(\frac{k^2}{j}\right)^k + e^{-j/2}j^k\end{aligned}$$

This estimate is then valid for all k.

In order to finish estimating (37) we need to deal with the terms involving $(s+1)$ rather than $(s-1)$. Observe that the pattern of differentiation is the same as for the $(s-1)$ terms, but on $[0,1]$ all the resulting terms are bounded by $e^{-j/2}$ because negative powers of $(s+1)$ are trivially bounded by 1. We conclude by the same method as above that

$$\left|\left(\frac{d}{ds}\right)^{j-k}\exp\left(\frac{j}{2(s+1)}\right)\right| \leq e^{-j/2}\, j^{(k-j)}$$

and can put all of our calculations together to conclude that

$$\begin{aligned}
C_j^{-1}\left|\left(\frac{d}{ds}\right)^{j}\chi_j(s)\right| &\leq \left|\sum_{k=0}^{j}\binom{j}{k}\cdot\left(\frac{d}{ds}\right)^{k}\exp\left(\frac{j}{2(s-1)}\right)\cdot\left(\frac{d}{ds}\right)^{j-k}\exp\left(\frac{-j}{2(s+1)}\right)\right| \\
&\leq \sum_{k=0}^{j}\binom{j}{k}e^{-j/2}j^{(k-j)}C^k\left(\frac{k^2}{j}\right)^{k} + \sum_{k=0}^{j}\binom{j}{k}e^{-j}j^j \\
&\leq j^je^{-j/2}\left[\sum_{k=0}^{j}\binom{j}{k}C^k\left(\frac{k}{j}\right)^{2k}j^{2(k-j)}\right] + 2^je^{-j}j^j \\
&\leq j^je^{-j/2}\left[\sum_{k=0}^{j}\binom{j}{k}C^kj^{2(k-j)}\right] + 2^je^{-j}j^j \\
&\leq j^je^{-j/2}\left(C+j^{-2}\right)^j + 2^je^{-j}j^j \\
&\leq j^je^{-j}\left(e^{j/2}(C+1)^j + 2^j\right)
\end{aligned}$$

Substituting into (36) and using Stirling's formula to estimate $j! \geq j^je^{-j}\sqrt{2\pi j}$ we have at last

$$\begin{aligned}
|\psi_j(r)| &\leq \frac{C_j j^j e^{-j}}{j^je^{-j}\sqrt{2\pi j}}\left(e^{j/2}(C+1)^j + 2^j\right)\left(\frac{2}{(r_{j+1}-r_j)}\right)^{j+1} \\
&\leq \left(\frac{c}{r_{j+1}-r_j}\right)^{j+1}
\end{aligned}$$

where we used the fact that $C_j \leq e^{4j/3}$. It is easily verified that we can take $c = 20$.

Construction

Beginning with ψ_0 we inductively subtract constant multiples of the functions ψ_j for $j \geq 1$ so that the resulting function on I has all its moments vanish except the one of zeroth order. The method serves as a model for our later construction of the function K in Lemma 17.

Call the function before the j-th stage of the induction Ψ_j and set $\Psi_0 = \psi_0$. The moments of Ψ_j are $a_k^j = \int_I r^k\Psi_j(r)\,dr$. It is clear that $a_k^0 = \mu_{0,k}$. In this notation the j-th stage of the induction is $\Psi_{j+1} = \Psi_j - a_{j+1}^j\psi_{j+1}$, from which the moments of Ψ_{j+1} are given by $a_k^{j+1} = a_k^j - a_{j+1}^j\mu_{j+1,k}$.

Observe that $a^{j+1}_{j+1} = 0$ because $\mu_{j+1,j+1} = 1$. Since $\mu_{l,j+1} = 0$ for all $l > j+1$ it follows that we have

$$a^{j+1}_k = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } 1 \le k \le j+1 \\ a^j_k - a^j_{j+1}\mu_{j+1,k} & \text{if } k > j+1 \end{cases}$$

as was intended. Each ψ_j is supported on the interval I_j and these intervals are disjoint, so it is apparent that to prove the $\Psi_j(r)$ converge all we need do is estimate the numbers a^j_{j+1} and use our estimates on the functions ψ_j. For this purpose we define a sequence $\{b^j_k\}$ by setting $b^0_k = |a^0_k| = |\mu_{0,k}|$ and $b^{j+1}_k = b^j_k + b^j_{j+1}|\mu_{j+1,k}|$. It is clear that $|a^0_k| \le b^0_k$ for all k. Assuming inductively that $|a^j_k| \le b^j_k$ we have

$$|a^{j+1}_k| \le |a^j_k| + |a_{j,j+1}|\mu_{j+1,k} \le b^j_k + b^j_{j+1}\mu_{j+1,k} = b^{j+1}_k \tag{39}$$

and henceforth need only consider the sequence $\{b^j_{j+1}\}$.

Estimates

Though we do not show it explicitly, the essential idea of the following estimates is that binomial factor in the $\mu_{j,k}$ causes terms to increase very rapidly as j and k increase (with $k > j$). This implies that at any stage of the induction the dominant terms will be from the moments of the most recently introduced ψ_j.

Lemma 19 *For $j \ge 1$ and $k \ge j$, the moments $\mu_{j,k}$ satisfy*

$$\frac{\mu_{j-1,k}}{\mu_{j-1,j}\mu_{j,k}} \le \frac{2}{k-j+1} \tag{40}$$

PROOF. We may use the fact that $\chi_{j-1}(s)$ is an even function on $[-1,1]$ to explicitly compute the term $\mu_{j-1,j}$.

$$\mu_{j-1,j} = \binom{j}{j-1}\left(\frac{r_j - r_{j-1}}{2}\right)\int_{-1}^{1}\left(s + \frac{r_j + r_{j-1}}{r_j - r_{j-1}}\right)\chi_{j-1}(s)\,ds = j\left(\frac{r_j - r_{j-1}}{2}\right)\left(\frac{r_j + r_{j-1}}{r_j - r_{j-1}}\right)$$

By the symmetry of $\phi_j(r)$ around the midpoint of I_j and the fact that r^{k-j} is an increasing function we have the bound

$$\mu_{j,k} = \binom{k}{j}\int_{I_j} r^{k-j}\phi_j(r)\,dr \ge \binom{k}{j}\left(\frac{r_{j+1} + r_j}{2}\right)^{k-j}$$

and we estimate $\mu_{j-1,k}$ using the upper endpoint of the interval:

$$\mu_{j-1,k} = \binom{k}{j-1}\int_{I_{j-1}} r^{k-j+1}\phi_{j-1}(r)\,dr \le \binom{k}{j-1} r_j^{k-j+1}.$$

Combining these we have

$$\begin{aligned}
\frac{\mu_{j-1,k}}{\mu_{j-1,j}\mu_{j,k}} &\le \frac{\binom{k}{j-1} r_j^{k-j+1}}{j\binom{k}{j}\left(\frac{r_j+r_{j-1}}{2}\right)\left(\frac{r_{j+1}+r_j}{2}\right)^{k-j}} \\
&= \frac{1}{k-j+1}\left(\frac{2r_j}{r_j+r_{j-1}}\right)\left(\frac{2r_j}{r_{j+1}+r_j}\right)^{k-j} \\
&\le \frac{2}{k-j+1}
\end{aligned}$$

Lemma 20 *The sequence b^j_{j+1} satisfies*

$$b^j_{j+1} \le e^2 b^{j-1}_j |\mu_{j,j+1}| \le e^{2j} \prod_{l=0}^{j} |\mu_{l,l+1}| \tag{41}$$

PROOF. We expand b^{j+1}_k from its definition to obtain

$$\begin{aligned}
b^{j+1}_k &= b^j_k + b^j_{j+1}|\mu_{j+1,k}| \\
&= b^{j-1}_k + b^{j-1}_j|\mu_{j,k}| + b^j_{j+1}|\mu_{j+1,k}| \\
&\vdots \\
&= b^0_k + b^0_1|\mu_{1,k}| + b^1_2|\mu_{2,k}| + \cdots + b^j_{j+1}|\mu_{j+1,k}| \\
&= |\mu_{0,k}| + b^0_1|\mu_{1,k}| + b^1_2|\mu_{2,k}| + \cdots + b^j_{j+1}|\mu_{j+1,k}|
\end{aligned} \tag{42}$$

and see that we must deal with a sum of terms of the type $b^{l-1}_l|\mu_{l,k}|$. Again applying the definition we have $b^{l-1}_l|\mu_{l,l+1}| = b^l_{l+1} - b^{l-1}_{l+1} \le b^l_{l+1}$, and in conjunction with the inequality (40) from the preceding lemma we obtain for $l \ge 1$

$$\begin{aligned}
b^{l-1}_l|\mu_{l,k}| &\le b^{l-1}_l|\mu_{l,l+1}||\mu_{l+1,k}|\left(\frac{2}{k-l}\right) \\
&\le b^l_{l+1}|\mu_{l+1,k}|\left(\frac{2}{k-l}\right) \\
&\vdots \qquad \text{inductively} \\
&\le b^j_{j+1}|\mu_{j+1,k}|\left(\frac{2^{j-l+1}}{(k-l)(k-l-1)\cdots(k-j)}\right)
\end{aligned}$$

The same method applies to estimate the first term in (42) because (40) implies $|\mu_{0,k}| \le \left(\frac{2}{k}\right)|\mu_{0,1}||\mu_{1,k}| = \left(\frac{2}{k}\right) b^0_1|\mu_{1,k}|$.

Now we need only substitute into the sum (42) to find that with $m = j - l$

$$b^{j+1}_k \le b^j_{j+1}|\mu_{j+1,k}|\left(1 + \sum_{m=0}^{j} \frac{2^{m+1}(k-j-1)!}{(k-j+m)!}\right)$$

and in particular

$$b^{j+1}_{j+2} \leq b^j_{j+1}|\mu_{j+1,j+2}|\left(1+\sum_{m=0}^{j}\frac{2^{m+1}}{(m+2)!}\right) \leq e^2 b^j_{j+1}|\mu_{j+1,j+2}|$$

which proves the first assertion of the lemma. The second follows from this using induction and the definition $b^0_1 = |\mu_{0,1}|$.

Properties of $\mathbf{\Psi(r)} = \lim \mathbf{\Psi_j(r)}$

Recall that the functions $\Psi_j(r)$ were defined inductively by

$$\Psi_0(r) = \psi_0(r), \qquad \Psi_{j+1}(r) = \Psi_j(r) - a^j_{j+1}\psi_{j+1}(r) \tag{43}$$

The functions $\psi_j(r)$ are defined on the disjoint intervals I_j, so it is immediate that the $\Psi_j(r)$ converge pointwise to a function $\Psi(r)$ on I that can be bounded by controlling $|a^j_{j+1}\psi_{j+1}|$. By (39), (41) and the fact that $|\mu_{l,l+1}| \leq (l+1)r_l$ from (34) we obtain

$$|a^j_{j+1}| \leq b^j_{j+1} \leq e^{2j}\prod_{l=0}^{j}|\mu_{l,l+1}| \leq e^{2j}(j+1)!\prod_{l=0}^{j} r_l.$$

Multiplying this by the bound for ψ_{j+1} we found in (35) yields for $r \in I_{j+1}$

$$|\Psi(r)| \leq |a^j_{j+1}|\,|\psi_{j+1}| \leq e^{2j}(j+1)!\left(\prod_{l=0}^{j} r_l\right)\left(\frac{20}{(r_{j+2}-r_{j+1})}\right)^{j+2}. \tag{44}$$

It is not hard to discover that the rate of growth of the sequence $\{r_j\}$ determines the bounds available from (44). The choice $r_j = R\exp\left[2\log^2(j+j_0)\right]$ from Lemma 17 is close to optimal, and we record the corresponding estimate as a lemma.

Lemma 21 *With* $\{r_j\}$ *as in Lemma 17 and* $j_0 \geq 8$ *we have*

$$j!\left(\prod_{l=0}^{j-1} r_l\right)\left(\frac{20}{(r_{j+1}-r_j)}\right)^{j+1} \leq \exp\Big(C + 2j_0\log^2(j+j_0) - 2(j+j_0)\log(j+j_0)\Big) \tag{45}$$

PROOF. For notational purposes it will be convenient for us to work with the logarithm of the above quantity. The relevant estimates are

$$
\begin{aligned}
&\log(r_{j+1} - r_j) \\
&\quad = \log\left[T\left(\exp(2\log^2(j+j_0))\right)\left(\exp(2\log^2(j+j_0+1) - 2\log^2(j+j_0))\right)\right] \\
&\quad \geq \log\left[T\left(\exp(2\log^2(j+j_0))\right)\left(2\log^2(j+j_0+1) - 2\log^2(j+j_0)\right)\right] \\
&\quad = \log T + 2\log^2(j+j_0) + \log 2 + \log\left[(\log(j+j_0+1)(j+j_0))\left(\log\left(1+\frac{1}{j+j_0}\right)\right)\right] \\
&\quad \geq \log T + 2\log^2(j+j_0) + \log 2 + \log(2\log(j+j_0)) + \log\log\left(1+\frac{1}{j+j_0}\right) \\
&\quad \geq \log T + 2\log^2(j+j_0) + \log 4 + \log\log(j+j_0) + \log\left(\frac{\log 2}{j+j_0}\right) \\
&\quad \geq \log T + 2\log^2(j+j_0) + \log\log(j+j_0) + \log(4\log 2) - \log(j+j_0) \qquad (46)
\end{aligned}
$$

and for the product term

$$
\begin{aligned}
\sum_{0}^{j-1} \log r_l &= j\log T + 2\sum_{0}^{j-1}\log^2(l+j_0) \\
&\leq j\log T + 2\int_{j_0}^{j+j_0}\log^2 x\,dx \\
&= j\log T + 2(j+j_0)\log^2(j+j_0) - 4(j+j_0)\log(j+j_0) \\
&\quad + 4(j+j_0) - 2j_0\log^2 j_0 + 4j_0\log j_0 - 4j_0 \qquad (47)
\end{aligned}
$$

Combining (46), (47), and the Stirling Estimate $j! \leq c\sqrt{j}\,j^j e^{-j}$ produces

$$
\begin{aligned}
&\log\left[j!\left(\prod_{l=0}^{j-1} r_l\right)\left(\frac{20}{(r_{j+1}-r_j)}\right)^{j+1}\right] \\
&\quad \leq \log c - j + (j+1/2)\log j + j\log T + 2(j+j_0)\log^2(j+j_0) \\
&\qquad - 4(j+j_0)\log(j+j_0) + 4j - 2j_0\log^2 j_0 + 4j_0\log j_0 \\
&\qquad - (j+1)\log T - 2(j+1)\log^2(j+j_0) - (j+1)\log\log(j+j_0) \\
&\qquad - (j+1)\log(4\log 2) + (j+1)\log(j+j_0) \\
&\quad \leq \log c + 2j_0\log^2(j+j_0) - 2(j+j_0)\log(j+j_0)
\end{aligned}
$$

because $j_0 \geq 8 \geq e^2$. Inserting the constant c for the Stirling estimate we obtain the conclusion of the lemma with $C = \log(\sqrt{2\pi}e)$.

Lemma 21 may be combined with (44) to produce an estimate valid on I_j

$$\begin{aligned}\log|\Psi(r)| &\leq \log(|a_j^{j-1}|\,|\psi_j|)\\ &\leq 2j-2+C+2j_0\log^2(j+j_0)-2(j+j_0)\log(j+j_0)\\ &\leq -(j+j_0+1)\log(j+j_0+1)\end{aligned}$$

for all sufficiently large j. However $\log r \leq \log T + 2\log^2(j+j_0+1)$ on I_j, so we see that for all sufficiently large values of r

$$\log|\Psi(r)| \leq -\left(\frac{1}{2}\log\frac{r}{T}\right)^{1/2}\exp\left(\frac{1}{2}\log\frac{r}{T}\right)^{1/2} \tag{48}$$

This is certainly sufficiently rapid decay to ensure integrability against the polynomials, and an application of the dominated convergence theorem shows

$$\int r^k\Psi(r)\,dr = \lim_{j\to\infty}\int r^k\Psi(r)\,dr = \begin{cases}1 & \text{if } k=0\\ 0 & \text{if } k=1,2,3,\ldots\end{cases} \tag{49}$$

so that we have found a smooth function with vanishing moments and exponential decay on the half line I. Our method is cruder than some of the known techniques, see for example Lemma 1 on page 182 of [18], and we pay a price in the rate at which the function decays. In compensation we have gained substantial control over the regions in which cancelation occurs for individual monomials.

Functions on Subsets of $\mathbf{S^{n-1}}$

The functions $\psi_j(r)$ can be used to select for the radial growth r^j, but in $\mathbb{R}^n$ there are many monomials with this rate of growth that need to be treated separately. This is achieved by constructing functions on a fixed subset of the unit sphere S^{n-1} with the property that they vanish when integrated against any monomial except the specific one desired. In our construction we work with angular variables rather than the restrictions of monomials to S^{n-1}.

Functions on an Arc of $\mathbf{S^1}$

Lemma 22 *Let Θ be an arc of angular length $|\Theta|$ in the unit circle S^1. For a fixed $J \in \mathbb{N}$ and for each $l \in \mathbb{Z}$ with $|l| < J$ there is a smooth function $G_l(\theta)$ with support* *in* Θ *such that*

$$\int_{S^1} G_l(\theta)e^{ik\theta}\,d\theta = \begin{cases} 1 & \textit{if } k = l \\ 0 & \textit{if } |k| \le J \textit{ and } k \neq l \end{cases} \tag{50}$$

$$|G_l(\theta)| \le \left(\frac{C}{|\Theta|}\right)^{2J+2} \tag{51}$$

PROOF. Without loss of generality we may identify Θ with the interval $[0, |\Theta|]$ in the angular variable. Let J and l be fixed.

Partition Θ using $\lambda_j = (2j+1)|\Theta|/(4J+2)$. For each $\phi \in [-|\Theta|/(4J+2), |\Theta|/(4J+2))$ consider also the partition translated by ϕ. Writing $z_j = e^{i\lambda_j}$ we define the Lagrange interpolating polynomials corresponding to these partitions

$$P_j(z) = \prod_{k=0,k\neq j}^{2J} \frac{z - z_k}{z_j - z_k}, \qquad P_{j,\phi}(z) = \prod_{k=0,k\neq j}^{2J} \frac{z - e^{i\phi}z_k}{e^{i\phi}z_j - e^{i\phi}z_k} = P_j(e^{-i\phi}z).$$

For all integers k with $|k| \le J$ we see that $e^{i(J+k)\theta}$ is a polynomial of degree at most $2J$ in $z = e^{i\theta}$, so it is determined by its values at the points of the partition and

$$e^{i(J+k)\theta} = z^{J+k} = \sum_{j=0}^{2J} (e^{i\phi}z_j)^{(J+k)} P_{j,\phi}(z) = \sum_{j=0}^{2J} e^{i(J+k)(\lambda_j+\phi)} P_j(e^{i(\theta-\phi)}).$$

Multiplying by $e^{-i(J+l)\theta}$ and integrating over $[0, 2\pi]$ we have

$$\int_0^{2\pi} e^{i(k-l)\theta}\,d\theta = \int_0^{2\pi} \sum_{j=0}^{2J} e^{i(J+k)(\lambda_j+\phi)} e^{-i(J+l)\theta} P_j(e^{i(\theta-\phi)})$$

so that setting

$$a_j(\phi) = \frac{e^{iJ(\lambda_j+\phi)}}{2\pi} \int_0^{2\pi} P_j(e^{i(\theta-\phi)})e^{-i(J+l)\theta}\,d\theta$$

we obtain

$$\sum_{j=0}^{2J} a_j(\phi)e^{ik(\lambda_j+\phi)} = \begin{cases} 1 & \text{if } k = l \\ 0 & \text{if } 0 \le |k| \le J \end{cases}$$

which may be viewed as the solution to a disctretized version of the problem on the partition $\{\lambda_j + \phi\}$. We can now complete the proof by integrating against a function $\eta(\phi) \in C^\infty$ that is supported on $[-|\Theta|/(4J+2), |\Theta|/(4J+2)]$. Write $\theta \in \Theta$ in its unique form $\theta = \lambda_j + \phi$ for ϕ in the given interval, and set $G_l(\theta) = a_j(\phi)\eta(\phi)$. This is a product of smooth functions on the intervals $(\lambda_j - |\Theta|/(4J+2), \lambda_j + |\Theta|/(4J+2))$, and at the points where these intervals meet we see that $\eta(\phi)$ and all its derivatives are zero, so G_l is smooth. Moreover

$$\begin{aligned}\int_\Theta G_l(\theta)e^{ik\theta}\,d\theta &= \sum_{j=0}^{2J}\int_{\lambda_j-|\Theta|/(4J+2)}^{\lambda_j+|\Theta|/(4J+2)} G_l(\theta)e^{ik\theta}\,d\theta \\ &= \sum_{j=0}^{2J}\int_{-|\Theta|/(4J+2)}^{|\Theta|/(4J+2)} a_j(\phi)\eta(\phi)e^{i(\lambda_j+\phi)k}\,d\phi \\ &= \begin{cases} 1 & \text{if } k=l \\ 0 & \text{if } 0\le |k|\le J\end{cases}\end{aligned}$$

With this definition of $G(\theta)$ it is easily verified that

$$|G_l(\theta)| \le \frac{\|\eta(\phi)\|_{L^\infty}}{2\pi}\int_0^{2\pi}|P_j(e^{i\lambda})|\,d\lambda \tag{52}$$

and since we may choose η with $|\eta(\phi)| \le C(2J+1)/|\Theta|$, we can establish (51) by estimating P_j. All terms in the numerator of P_j are bounded individually by 2 for z on the unit circle, and the denominator is clearly largest for the case $j = J+1$ when we obtain

$$\prod_{k=0,k\neq j}^{2J}(z_j-z_k) = \left(\frac{|\Theta|}{4J+2}\right)^{2J+1}(J!)^2 \ge \left(\frac{|\Theta|}{4J+2}\right)^{2J+1} 2\pi J^{2J+1}e^{-2J} \ge 2\pi e\left(\frac{|\Theta|}{6e}\right)^{2J+1}$$

where we used that $J! > \sqrt{2\pi J}J^Je^{-J}$ and $J/(2J+1) \ge 1/3$. From these and (52)

$$|G_l(\theta)| \le \frac{C(2J+1)}{4\pi^2 e|\Theta|}\left(\frac{12e}{|\Theta|}\right)^{2J+1} \le \left(\frac{C}{|\Theta|}\right)^{2J+2}$$

Functions on subsets of $\mathbf{S^{n-1}}$

We use the coordinate system $(x_1,\dots,x_n)$ on $\mathbb{R}^n$ to define generalized spherical coordinates $(\theta_1,\dots,\theta_{n-1})$ on the unit sphere S^{n-1} according to

$$\xi_j = \begin{cases} \cos\theta_1 & \text{if } j=1 \\ \cos\theta_j \prod_{k=1}^{j-1}\sin\theta_k & \text{if } 1<j<n \\ \prod_{k=1}^{n-1}\sin\theta_k & \text{if } j=n\end{cases}$$

Notice that $\theta_j \in [0,\pi]$ for $j < n-1$ while $\theta_{n-1} \in [0,2\pi)$, and that the Jacobian relating the new coordinates to the old is $\mathcal{J} = \prod_{k=1}^{n-2}\sin^{n-k-1}\theta_k$.

Suppose we have an angular cube, i.e. a set of the form $\{(\theta_1,\dots,\theta_{n-1}) : \theta_j \in \Theta_j\}$ with each Θ_j an arc of the same angular length. We use $|\Theta|$ for the length of the cube. For $J \in \mathbb{N}$ and a multi-index $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$ with all $|\alpha_j| \leq J$, let $G_{\alpha_j}(\theta)$ be as in Lemma 22. The product of these functions has the properties

$$\int_{S^{n-1}} \left(\prod_{j=1}^{n-1} G_{\alpha_j}(\theta_j)\right)\left(\prod_{j=1}^{n-1} e^{i\alpha_j\theta_j}\right) d\theta_1 \cdots d\theta_{n-1} = \prod_{j=1}^{n-1} \int_0^{\pi} G_{\alpha_j}(\theta_j) e^{i\alpha_j\theta_j}\, d\theta_j$$
$$= \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if some } \beta_j \leq J \text{ and } \alpha \neq \beta \end{cases}$$
$$\left|\prod_{j=1}^{n-1} G_{\alpha_j}(\theta_j)\right| \leq \prod_{j=1}^{n-1} \left(\frac{C}{|\Theta|}\right)^{2(J+1)} = \left(\frac{C}{|\Theta|}\right)^{2(n-1)(J+1)}$$

In what follows we wish to integrate with respect to the restriction $d\sigma(x)$ of $dx_1 \ldots dx_n$ to S^{n-1} rather than with respect to the angular variables, for which reason we define

$$H_\alpha = \frac{1}{\mathcal{J}} \prod_{j=1}^{n-1} G_{\alpha_j}(\theta_j)$$

It is not difficult to show that a set of the form $B(\xi, \lambda) \cap S^{n-1}$ supports functions of this type, and that we may assume $\mathcal{J} \geq C\lambda^{n-2}$. Observe first that

$$|\mathcal{J}| = \prod_{k=1}^{n-2} |\sin\theta_k|^{n-k-1} \geq \left(\prod_{k=1}^{n-2} |\sin\theta_k|\right)^{n-2} = \left(\xi_{n-1}^2 + \xi_n^2\right)^{(n-2)/2} \tag{53}$$

and that $\{\xi_{n-1}^2 + \xi_n^2 \geq c_1\lambda^2\} \cap (B(\xi, \lambda) \cap S^{n-1}) \supset B(\tilde{\xi}, c_2\lambda) \cap S^{n-1}$ for some absolute constants c_1 and c_2. This latter set clearly contains an angular cube of length at least $c_3(n)\lambda$ and we obtain the bound on $\mathcal{J}$ from (53). We summarize our findings as a lemma.

Lemma 23 *Let $\Lambda = B(\xi, \lambda) \cap S^{n-1}$ where $\xi \in S^{n-1}$ and $\lambda < 1$. Fix $J \in \mathbb{N}$ and let $\alpha = (\alpha_1, \ldots, \alpha_n)$ satisfy $|\alpha_j| \leq J$ for all j. Then there is $H_\alpha \in C^\infty(S^{n-1})$ and supported on Λ such that*

$$\int_{S^{n-1}} H_\alpha(x) \exp\left(i \sum_{j=1}^{n-1} \beta_j\theta_j\right) d\sigma(x) = \begin{cases} 1 & \textit{if } \beta = \alpha \\ 0 & \textit{if some } |\beta_j| \leq J \textit{ and } \beta \neq \alpha \end{cases} \tag{54}$$
$$|H_\alpha| \leq \left(\frac{C}{\lambda}\right)^{(n-1)(2J+3)}$$

Building Blocks and Bounds

The hypothesis of Lemma 17 gives $\Gamma = \cup\Gamma_j$ where

$$\Gamma_j = \left\{ r_j \leq |x| \leq r_{j+1}, \frac{x}{|x|} \in \Lambda_j \right\} \quad \text{and} \quad \Lambda_j = S^{n-1} \cap B(\xi_j, \lambda)$$

with λ independent of j. For each j and multi-index α with all $|\alpha_j| \leq j$ we set $J = 2j + 2$ and apply Lemma 23 on Λ_j to define functions $H_{j,\alpha}$. Let the functions $\psi_j(r)$ be as in (32) and set

$$F_{(j,\alpha)}(r, \xi) = \psi_j(r) H_{j,\alpha}(\xi).$$

These functions are C^∞, supported on Γ_j, and by Lemmas 18 and 23 we have

$$\left| F_{(j,\alpha)}(r, \xi) \right| \leq \left(\frac{C}{\lambda} \right)^{(n-1)(4j+7)} \left(\frac{20}{r_{j+1} - r_j} \right)^{j+1} \tag{55}$$

If we denote the moments with respect to the functions $r^k e^{i\beta\theta}$ by

$$M_{(j,\alpha),(k,\beta)} = \int_{\mathbb{R}^n} F_{(j,\alpha)}(r, \theta) r^k e^{i\beta\theta} \, d\sigma \, dr$$

then we derive from (33) and Lemma 23 that

$$M_{(j,\alpha),(k,\beta)} = \begin{cases} 0 & \text{if some } |\beta_l| \leq 2j + 2 \text{ and } \beta \neq \alpha, \text{ or if } k < j \\ 1 & \text{if } \beta = \alpha \text{ and } k = j \\ \mu_{j,k} & \text{if } \beta = \alpha \text{ and } k > j \end{cases} \tag{56}$$

In the remaining case where all $|\beta_l| \geq 2j + 3$ and $k \geq j$ we have from (23) that

$$\left| M_{(j,\alpha),(k,\beta)} \right| \leq \mu_{j,k} \left(\frac{C}{\lambda} \right)^{(n-1)(4j+7)} \tag{57}$$

however in what follows we will only be interested in those moments $M_{(j,\alpha),(k,\beta)}$ for which $k \geq \max_l |\beta_l|$. For these we use $k \geq 2j + 3$ to rewrite (57) as

$$\left| M_{(j,\alpha),(k,\beta)} \right| \leq \mu_{j,k} \left(\frac{C}{\lambda} \right)^{4(n-1)(k-j-1)} . \tag{58}$$

Construction

As in the one dimensional case we inductively construct a function with vanishing moments. Set $K^0(r,\theta) = F_{0,0}(r,\theta)$ and define

$$N^j_{(k,\beta)} = \int_{\mathbb{R}^n} K^j(r,\theta) r^k e^{i\beta\theta}\, d\sigma\, dr \tag{59}$$

$$K^{j+1}(r,\theta) = K^j(r,\theta) - \sum_{l=1}^{n-1} \sum_{|\alpha_l|\le j+1} N^j_{(j+1,\beta)} F_{(j+1,\alpha)}(r,\theta) \tag{60}$$

so that $N^{j+1}_{(j+1,\beta)} = 0$ for all β satisfying $|\beta_l| \le j+1$, $l = 1,\ldots,n-1$. By (56) the functions $F_{(j+1,\alpha)}$ do not affect the moments $N^{j+1}_{(k,\beta)}$ for $k \le j$, and consequently

$$N^{j+1}_{(k,\beta)} = \begin{cases} 1 & \text{if } k = 0 \text{ and } \beta = (0,\ldots,0) \\ 0 & \text{if } k \le j+1 \text{ and } |\beta_l| \le j+1 \text{ for } l = 1,\ldots,n-1 \end{cases} \tag{61}$$

There are finitely $F_{j,\alpha}$ for each j, all of which are supported on Γ_j. Since the sets Γ_j are disjoint the functions $K^j(x)$ have a pointwise limit function $\tilde{K}(x)$ supported on Γ. We show this limit is integrable against polynomials and has vanishing moments.

Estimates

Our model is the estimation scheme for the one dimensional case. Notice that the moment sequence $N^j_{(k,\beta)}$ evolves according to the induction

$$N^{j+1}_{(k,\beta)} = N^j_{(k,\beta)} - \sum_{l=1}^{n-1} \sum_{|\alpha_l|\le j+1} N^j_{(j+1,\alpha)} M_{(j+1,\alpha),(k,\beta)} \tag{62}$$

We are only interested in moments (k,β) for which $k \ge \max_l |\beta_l|$. In this situation we may compare (56) and (58) to see that all of the moments $M_{(j+1,\alpha),(k,\beta)}$ occurring in the sum satisfy

$$\left| M_{(j+1,\alpha),(k,\beta)} \right| \le \mu_{j+1,k} \left(\frac{C}{\lambda} \right)^{4(n-1)(k-j-2)} \tag{63}$$

It is also easily seen that the number of terms in this sum is $(2j+3)^{n-1}$. These observations suggest defining a new sequence by

$$P^0_k = \max \left\{ \left| M_{(0,0),(k,\beta)} \right| : |\beta_l| \le k \text{ for all } l = 1,\ldots,n-1 \right\} \tag{64}$$

$$P^{j+1}_k = P^j_k + P^j_{j+1} \mu_{j+1,k} \left(\frac{C_0}{t} \right)^{4(n-1)(k-j-2)} \tag{65}$$

where $C_0 = 2C$ is twice the constant in (63) and is fixed from here onward. Our previous work shows that C_0 depends only upon the dimension n.

The benefit of this new sequence is that it dominates the sequence $N^j_{(k,\beta)}$ but will be much simpler to analyze. We record this as a lemma.

Lemma 24 *For all j, k, and β with $|\beta_l| \le k$, $l = 0, \ldots, n-1$ we have $\left|N^j_{(k,\beta)}\right| \le P^j_k$.*

PROOF. For $j = 0$ this is obvious from the definition. Assuming the truth of the estimate for all superindices up to j we proceed inductively, looking at two cases. If $k \le 2j+4$ then $|\beta_l| \le 2j+4$ and so by (56) all $M_{(j+1,\alpha),(k,\beta)} = 0$. Therefore

$$\left|N^{j+1}_{(k,\beta)}\right| = \left|N^j_{(k,\beta)} - \sum_{l=1}^{n-1} \sum_{|\alpha_l| \le j+1} N^j_{(j+1,\alpha)} M_{(j+1,\alpha),(k,\beta)}\right| = \left|N^j_{(k,\beta)}\right| \le P^j_k \le P^{j+1}_k$$

If $k \ge 2j+5$ we use the bound (63) to obtain

$$\begin{aligned}
\left|N^{j+1}_{(k,\beta)}\right| &= \left|N^j_{(k,\beta)} - \sum_{l=1}^{n-1} \sum_{|\alpha_l| \le j+1} N^j_{(j+1,\alpha)} M_{(j+1,\alpha),(k,\beta)}\right| \\
&\le \left|N^j_{(k,\beta)}\right| + \left|\sum_{l=1}^{n-1} \sum_{|\alpha_l| \le j+1} N^j_{(j+1,\alpha)}\right| \mu_{j+1,k} \left(\frac{C}{\lambda}\right)^{4(n-1)(k-j-2)} \\
&\le P^j_k + (2j+3)^{n-1} P^j_{j+1} \mu_{j+1,k} \left(\frac{C}{\lambda}\right)^{4(n-1)(k-j-2)} \\
&\le P^j_k + P^j_{j+1} \mu_{j+1,k} \left(\frac{C_0}{\lambda}\right)^{4(n-1)(k-j-2)} \\
&= P^{j+1}_k
\end{aligned}$$

In the last step we used that $k \ge 2j+5$ whence $4(k-j-2) \ge 4j+12$ and so $(2j+3)^{n-1}$ is certainly dominated by $2^{(n-1)(4j+12)} = 2^{4(n-1)(k-j-2)}$.

Our estimates for $\{P^j_k\}$ closely mimic those for the one dimensional case. The key result is

Lemma 25 *The off-diagonal terms of the sequence $\{P^j_k\}$ satisfy the estimate*

$$P^j_{j+1} \le C \frac{e^{2A(j-1)}}{A^{j-8}} \prod_{l=0}^{j} \mu_{l,l+1} \tag{66}$$

where $A = \left(\frac{C_0}{\lambda}\right)^{4(n-1)}$ and C is independent of n and λ.

PROOF. Expanding P_k^{j+1} from the definition (65) we have

$$\begin{aligned} P_k^{j+1} &= P_k^j + P_{j+1}^j \mu_{j+1,k} A^{(k-j-2)} \\ &= P_k^{j-1} + P_j^{j-1} \mu_{j,k} A^{(k-j-1)} + P_{j+1}^j \mu_{j+1,k} A^{(k-j-2)} \\ &\vdots \qquad\qquad (67) \\ &= P_k^0 + P_1^0 \mu_{1,k} A^{k-2} + P_2^1 \mu_{2,k} A^{k-3} + \cdots + P_{j+1}^j \mu_{j+1,k} A^{(k-j-2)} \qquad (68) \end{aligned}$$

From (65) we see $P_{l+1}^l = P_{l+1}^{l-1} + P_l^{l-1}\mu_{l,l+1}$ whence $P_l^{l-1}\mu_{l,l+1} \le P_{l+1}^l$. Using (40) and this repeatedly we estimate the general term of (68)

$$\begin{aligned} P_l^{l-1}\mu_{l,k} &\le \left(\frac{2}{k-l}\right) P_l^{l-1} \mu_{l,l+1}\mu_{l+1,k} \\ &\le \left(\frac{2}{k-l}\right) P_{l+1}^l \mu_{l+1,k} \\ &\vdots \\ &\le \left(\frac{2}{k-l}\right)\left(\frac{2}{k-l-1}\right)\cdots\left(\frac{2}{k-j}\right) P_{j+1}^j \mu_{j+1,k} \\ &= \frac{(k-j-1)!2^{(j-l+1)}}{(k-l)!} P_{j+1}^j \mu_{j+1,k} \qquad (69) \end{aligned}$$

It is also straightforward from (56), (57), and (40) to see that

$$\begin{aligned} P_k^0 &= \max\left\{\left|M_{(0,0),(k,\beta)}\right| : |\beta_l| \le k \text{ for all } l = 1,\ldots,n-1\right\} \\ &\le A^7 \left(\frac{2}{k}\right) P_1^0 \mu_{1,k} \end{aligned}$$

so that applying (69) for the case $l = 1$ we have

$$P_k^0 \le A^7 \frac{(k-j-1)!2^{(j+1)}}{k!} P_{j+1}^j \mu_{j+1,k} \qquad (70)$$

Now we may substitute the estimates (69) and (70) into the expression (68) for P_k^{j+1} and obtain

$$\begin{aligned} P_k^{j+1} &= P_k^0 + \sum_{l=1}^{j+1} P_l^{l-1}\mu_{l,k} A^{(k-l-1)} \\ &\le \left[A^7 \frac{(k-j-1)!2^{(j+1)}}{k!} + \sum_{l=1}^{j+1} \frac{(k-j-1)!2^{(j-l+1)}}{(k-l)!} A^{(k-l-1)} \right] P_{j+1}^j \mu_{j+1,k} \end{aligned}$$

We only need this result for the case $k = j + 2$ where it reduces to

$$
\begin{aligned}
P^{j+1}_{j+2} &\le \left[\frac{A^7 2^{(j+1)}}{(j+2)!} + \sum_{l=1}^{j+1} \frac{(2A)^{(j-l+1)}}{(j+2-l)!}\right] P^j_{j+1}\mu_{j+1,j+2} \\
&= \left[\frac{A^7 2^{(j+1)}}{(j+2)!} + \frac{1}{2A}\sum_{m=1}^{j+1} \frac{(2A)^m}{m!}\right] P^j_{j+1}\mu_{j+1,j+2} \\
&\le \begin{cases} \frac{1}{2A} e^{2A} P^j_{j+1}\mu_{j+1,j+2} & \text{if } j \ge 6 \\ \left(A^7 + \frac{1}{2A}e^{2A}\right) P^j_{j+1}\mu_{j+1,j+2} & \text{if } j < 6 \end{cases}
\end{aligned}
$$

Providing $A \ge 10$ the above factor is bounded by (e^{2A}/A) independently of j, so inserting a small constant to resolve this case we can inductively reduce to

$$
P^{j+1}_{j+2} \le C\frac{e^{2Aj}}{A^j} P^1_0 \prod_{l=1}^{j+1} \mu_{l,l+1} \le C\frac{e^{2Aj}}{A^{j-7}} \prod_{l=0}^{j+1} \mu_{l,l+1}
$$

Properties of the Kernel

On Γ_{j+1} we use (60) and the fact that the only non-zero $F_{(l,\alpha)}(r,\xi)$ have $l = j + 1$ to see that

$$
\tilde{K}(x) = -\sum_{l=1}^{n-1} \sum_{|\alpha_l| \le j+1} N^j_{(j+1,\beta)} F_{(j+1,\alpha)}(r,\xi)
$$

By (24) this implies $\left|\tilde{K}(x)\right| \le (2j+3)^{n-1} P^j_{j+1} \left|F_{(j+1,\alpha)}(r,\xi)\right|$, so that substituting the bounds (55) and (66) and writing both in terms of A, then using (34) gives

$$
\begin{aligned}
\left|\tilde{K}(x)\right| &\le C(2j+3)^{n-1}\frac{e^{2A(j-1)}}{A^{j-8}}\left(\frac{A}{2^{4(n-1)}}\right)^{j+1}\left(\frac{20}{r_{j+2}-r_{j+1}}\right)^{j+2}\prod_{l=0}^{j}\mu_{l,l+1} \\
&\le \frac{C}{A^7}e^{2A(j-1)}\left(\frac{20}{r_{j+2}-r_{j+1}}\right)^{j+2}\prod_{l=0}^{j}(l+1)r_l \\
&= \frac{C}{A^7}e^{2A(j-1)}(j+1)!\left(\prod_{l=0}^{j} r_l\right)\left(\frac{20}{r_{j+2}-r_{j+1}}\right)^{j+2}
\end{aligned}
$$

This is now very similar to the situation encountered in our one dimensional construction. With $r_j = T\exp\left[2\log^2(j + j_0)\right]$ we can directly apply Lemma 21 to obtain for $x \in \Gamma_j$

$$
\begin{aligned}
\log\left|\tilde{K}(x)\right| &\le C - 7\log A + 2A(j-2) + 2j_0\log^2(j+j_0) - 2(j+j_0)\log(j+j_0) \\
&\le -(j+j_0+1)\log(j+j_0+1) \qquad (71)
\end{aligned}
$$

provided j is sufficiently large. As $\log|x| \le \log T + 2\log^2(j + j_0 + 1)$ on Γ_j, we obtain

$$\log\left|\tilde{K}(x)\right| \le -\left(\frac{1}{2}\log\frac{|x|}{T}\right)^{1/2}\exp\left(\frac{1}{2}\log\frac{|x|}{T}\right)^{1/2} \tag{72}$$

for all sufficiently large $|x|$. This rate of decay ensures $\tilde{K}(x)$ is integrable against all functions having at most polynomial growth in the variable $|x|$, and by (61) and the dominated convergence theorem we have

$$\int_{\mathbb{R}^n} \tilde{K}(r,\xi) r^k e^{i\beta\theta}\, d\sigma(\theta)\, dr = \begin{cases} 1 & \text{if } k = 0 \text{ and } \beta = (0,\ldots,0) \\ 0 & \text{if } k \in \mathbb{N}\setminus\{0\} \text{ and all } |\beta_l| \le k \end{cases} \tag{73}$$

Since any monomial x^α may be written

$$x^\alpha = r^{|\alpha|}\sum_\beta a_\beta e^{i\beta\theta}$$

where $r = |x|$ and each β occurring in the sum satisfies $|\beta_l| \le |\alpha|$ for $l = 1, 2, \ldots, n$, we see that $\tilde{K}$ has vanishing moments of all orders. As x^α is real-valued the same is true of the real part $\mathrm{Re}(\tilde{K})$, so defining

$$K(x) = \frac{\mathrm{Re}(\tilde{K}(x))}{|x|^{n-1}}$$

we have that $K \in C^\infty(\mathbb{R}^n)$ is supported on Γ and

$$\int_{\mathbb{R}^n} K(x)x^\alpha\, dx = \int_{\mathbb{R}^n} \mathrm{Re}(\tilde{K}(x))x^\alpha d\sigma\, dr = \begin{cases} 1 & \text{if } \alpha = (0,\ldots,0) \\ 0 & \text{if } \alpha \in \mathbb{N}^n\setminus\{(0,\ldots,0)\} \end{cases}$$

Comparing (72) with (30) we find that $|K(x)| \le |x|^{1-n}\kappa(|x|)$. This completes the proof of Lemma 17 and therefore Theorem 16.

4 Extension on a Whitney Cube

Given $f \in L^p_k(\Omega)$ and $Q \in \mathcal{W}_2$ we define a function $\mathcal{E}_Q f$ on $(17/16)Q$ and identify some of its properties.

Definition of the extension

Let $\phi(x)$ be a C^∞ cutoff function such that $\phi \equiv 1$ on $\{\mathrm{dist}(x,\partial\Omega) \le \lambda\}$ and $\phi \equiv 0$ on $\{\mathrm{dist}(x,\partial\Omega) \ge 2\lambda\}$, where λ depends only on n, ϵ and δ. It is clear from the Leibnitz rule that

$$\|\phi f\|_{L^p_k(\Omega)} \le C(n,\epsilon,\delta)\|f\|_{L^p_k(\Omega)}$$

for all $f \in L^p_k(\Omega)$. Moreover an extension of ϕf to $\mathbb{R}^n \setminus \Omega$ also extends f. It follows that to prove Theorem 8 we need only treat the functions supported near $\partial\Omega$. In particular we henceforth assume that $f \equiv 0$ on all Whitney cubes $S \in \mathcal{W}_1$ with $l(S) \geq \epsilon\delta/(100\sqrt{n})$.

Denote the Whitney cubes from $\mathcal{W}_2$ with $l(Q) \leq \epsilon\delta/(200n)$ by $\mathcal{W}_3$. On these we will define $\mathcal{E}_Q(f)$ by convolution against a function of the type in Theorem 16, but first we need some preliminaries.

Corresponding to Q we have a chain of cubes $\{S_j\}$ as in Lemma 12 and a twisting cone Γ_Q contained in $\cup S_j$. We translate the center x_Q of Q to the origin and rescale by $(l(Q))^{-1}$, using tildes to indicate the scaled quantities. For example $\tilde{\Gamma}_Q = (l(Q))^{-1}(\Gamma_Q - x_Q)$ is a twisting cone centered at the origin.

From (10) we see that at distance $\tilde{r}$ from the origin there is $\tilde{y}$ with $|\tilde{y}| = \tilde{r}$ and $B(\tilde{y}, \eta|\tilde{y}|) \subset \tilde{\Gamma}_Q$ provided $\tilde{r} \in [R_0, R_1(l(Q))^{-1}]$, where R_0, R_1 and η depend only on n, ϵ and δ, and we can take $R_1 = \epsilon\delta/10$. By adjoining a piece of a cone to $\tilde{\Gamma}_Q$ we can make this property true for all $\tilde{r} \geq R_0$. Let $B(\tilde{y}, \eta|\tilde{y}|)$ be the ball in $\tilde{\Gamma}_Q$ at radius $|\tilde{y}| = R_1(l(Q))^{-1}$ and define

$$\tilde{\Gamma}^*_Q = \left(\tilde{\Gamma}_Q \cap \left\{R_0 \leq |\tilde{x}| \leq \frac{R_1}{l(Q)}\right\}\right) \cup \left\{\tilde{x} : |\tilde{x}| \geq \frac{R_1}{l(Q)} \text{ and } \frac{R_1 x}{|\tilde{x}| l(Q)} \in B(\tilde{y}, \eta|\tilde{y}|)\right\}$$

In keeping with our tilde notation we have $\Gamma^*_Q = l(Q)(\tilde{\Gamma}^*_Q + x_Q)$, and the result of this construction is shown in Figure 4.

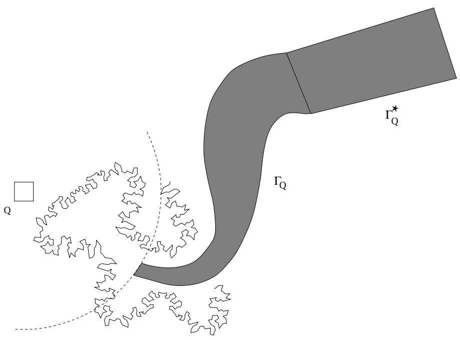

Fig. 2. The set Γ^*_Q

We record a trivial consequence of Lemma 13.

Lemma 26 *If $\tilde{y} \in \tilde{\Gamma}_Q$ is such that $(x_Q + l(Q)\tilde{y}) \in \Gamma_Q \cap S_j$, then for any $x \in (17/16)Q$ we have $(x + l(Q)\tilde{y}) \in S_{j-1} \cup S_j \cup S_{j+1}$.*

Now Theorem 16 applies to $\tilde{\Gamma}^*_Q$ so we have a smooth function $\tilde{K}_Q(\tilde{y})$ supported on

$\tilde{\Gamma}^*_Q$ with $|\tilde{K}_Q(\tilde{y})| \le \kappa(|\tilde{y}|)|\tilde{y}|^{1-n}$ and vanishing moments

$$\int_{\mathbb{R}^n} \tilde{y}^\alpha \tilde{K}_Q(\tilde{y}) = \begin{cases} 1 & \text{if } \alpha = (0,\dots,0) \\ 0 & \text{if } \alpha \in \mathbb{N}^n \setminus \{(0,\dots,0)\} \end{cases} \tag{74}$$

where C and T depend only on n, ϵ and δ. Notice that if $x \in (17/16)Q$ and $y \in S_j$ then by Lemma 26 and the linear growth (10) we have

$$\left|\tilde{K}_Q\left(\frac{y-x}{l(Q)}\right)\right| \le \left(\frac{l(Q)}{l(S_j)}\right)^{n-1} \kappa\left(\frac{l(S_j)}{l(Q)}\right). \tag{75}$$

We wish to define $\mathcal{E}_Q f$ as a convolution of f and K_Q, but must first arrange that f is defined on all of Γ^*_Q. This is done by setting

$$f_Q(x) = \begin{cases} f(x) & \text{if } |x - x_Q| \le R_1 \\ 0 & \text{otherwise} \end{cases} \tag{76}$$

which is a smooth continuation of f from Γ_Q to Γ^*_Q because the Whitney cubes that intersect Γ_Q at radius R_1 have length at least $\epsilon\delta/(10\sqrt{n})$ and therefore $f \equiv 0$ there by assumption. Now for $x \in (17/16)Q$ let

$$\mathcal{E}_Q f(x) = \begin{cases} \displaystyle\int_{\mathbb{R}^n} f_Q(x + l(Q)\tilde{y})\tilde{K}_Q(\tilde{y})\, d\tilde{y} & \text{if } Q \in \mathcal{W}_3 \\ 0 & \text{if } Q \in \mathcal{W}_2 \setminus \mathcal{W}_3 \end{cases} \tag{77}$$

By Lemma 26 this only involves the values of f_Q on a subset of $\cup S_j$ where we know $f_Q \equiv f$. In particular it would suffice to integrate over $\tilde{y} \in \tilde{\Gamma}_Q$ because $f_Q \equiv 0$ when $\tilde{y} \in \tilde{\Gamma}^*_Q \setminus \tilde{\Gamma}_Q$, so for $Q \in \mathcal{W}_3$ we may write

$$\mathcal{E}_Q f(x) = \int_{\tilde{\Gamma}_Q} f(x + l(Q)\tilde{y})\tilde{K}_Q(\tilde{y})\, d\tilde{y}. \tag{78}$$

Useful Estimates for $\mathbf{K_Q}$

To assist in the flow of the proof and avoid repetition we list some estimates for sums and integrals of $\tilde{K}_Q$.

Lemma 27 *With $\kappa(t)$ as defined in* (30) *we have $C = C(n, \epsilon, \delta, q)$ such that*

$$\sum_{j=m}^{\infty} 2^{qj}\kappa(2^j) \le C\, 2^{qm}\kappa(2^m)$$

PROOF. By (30) there are constants c_1, c_2, and c_3 depending only on n, ϵ and δ, such that

$$\begin{aligned}\sum_{j=m}^{\infty} 2^{qj}\kappa(2^j) &\leq 2^{qm}\kappa(2^m)c_1 \int_m^{\infty} \exp\left[c_2 q(t-m) - c_3\left(t^{1/2}e^{c_3 t^{1/2}} - m^{1/2}e^{c_3 m^{1/2}}\right)\right] dt \\ &= 2^{qm}\kappa(2^m)c_1 \int_0^{\infty} \exp\left[c_2 qs - c_3\left((s+m)^{1/2}e^{c_3 (s+m)^{1/2}} - m^{1/2}e^{c_3 m^{1/2}}\right)\right] ds \\ &= 2^{qm}\kappa(2^m)I(m,q)\end{aligned}$$

where $I(m,q)$ is finite for any $m \geq 0$ and $q \in \mathbb{R}$ and depends continuously on m. However if $m > c_3^{-2}$ then convexity implies

$$c_3(s+m)^{1/2}e^{c_3(s+m)^{1/2}} - c_3 m^{1/2}e^{c_3 m^{1/2}} \geq c_3 s e^{c_3 s^{1/2}} - e$$

so that in this case $I(m,q) \leq e^e I(0,q)$ and the result follows with C equal to the larger of $e^e I(0,q)$ and the maximum of $I(m,q)$ over $m \in [0, c_3^{-2}]$.

Corollary 28

$$\int_{\mathbb{R}^n} \left|\tilde{K}_Q(\tilde{y})\right| d\tilde{y} \leq C(n,\epsilon,\delta)$$

PROOF. Integrate radially by dividing $\mathbb{R}^n$ into concentric annuli from radius 2^j to 2^{j+1}. As $|\tilde{K}_Q(\tilde{y})| \leq \kappa(|\tilde{y}|)|\tilde{y}|^{1-n}$ and is supported on $[R_0, \infty)$, where R_0 depends on n, ϵ and δ, we see that

$$\int_{\mathbb{R}^n} \left|\tilde{K}_Q(\tilde{y})\right| d\tilde{y} \leq C(n,\epsilon,\delta)\sum_{j=0}^{\infty} 2^{j(1-n)}\kappa(2^{j+1})$$

and the result follows from Lemma 27.

Estimates for Individual Cubes

The following lemma allows control of the behavior of $\mathcal{E}_Q$ on the cube Q.

Lemma 29 *There are constants $C = C(n,\epsilon,\delta,k,p)$ such that*

$$\sum_{Q \in \mathcal{W}_2} \|D^\alpha \mathcal{E}_Q f\|^p_{L^p(Q)} \leq C\left\|D^\alpha f(z)\right\|^p_{L^p(\Omega)} \qquad \textit{if } 1 \leq p < \infty \tag{79}$$

$$\|D^\alpha \mathcal{E}_Q f\|_{L^\infty(Q)} \leq C\|D^\alpha f\|_{L^\infty(\Omega)} \qquad \textit{if } p = \infty \tag{80}$$

PROOF. The estimate is trivial for those cubes where $\mathcal{E}_Q$ is identically zero, so we may restrict our attention to $Q \in \mathcal{W}_3$. As f and its derivatives are locally integrable and $\tilde{K}_Q$ has rapid decay we may differentiate within the integral (78) to obtain

$$D^\alpha \mathcal{E}_Q f(x) = \int_{\tilde{\Gamma}_Q} D^\alpha f(x + l(Q)\tilde{y})\tilde{K}_Q(\tilde{y})\, d\tilde{y} \tag{81}$$

Applying Corollary 28 to $f \in L^p_k(\Omega)$ disposes of the case $p = \infty$.

$$\begin{aligned} \left|D^\alpha \mathcal{E}_Q f(x)\right| &\le \left\|D^\alpha f\right\|_{L^\infty(\Omega)} \int_{\mathbb{R}^n} \left|\tilde{K}_Q(\tilde{y})\right| d\tilde{y} \\ &\le C \left\|D^\alpha f\right\|_{L^\infty(\Omega)} \end{aligned}$$

with a constant $C = C(n, \epsilon, \delta)$. For the remainder of the proof we will therefore assume that $1 \le p < \infty$.

By Hölder's inequality and Corollary 28 applied to (81) we obtain after a change of variables

$$\begin{aligned} \left|D^\alpha \mathcal{E}_Q f(x)\right|^p &\le \left(\int_{\tilde{\Gamma}_Q} \left|D^\alpha f(x + l(Q)\tilde{y})\right|^p \left|\tilde{K}_Q(\tilde{y})\right| d\tilde{y}\right) \left(\int_{\mathbb{R}^n} \left|\tilde{K}_Q(\tilde{y})\right| d\tilde{y}\right)^{p-1} \\ &\le C \int_{\tilde{\Gamma}_Q} \left|D^\alpha f(x + l(Q)\tilde{y})\right|^p \left|\tilde{K}_Q(\tilde{y})\right| d\tilde{y} \\ &= \frac{C}{l(Q)^n} \int_{\cup S_j} \left|D^\alpha f(z)\right|^p \left|\tilde{K}_Q\left(\frac{z - x}{l(Q)}\right)\right| dz. \end{aligned}$$

Using (75) to estimate $\left|\tilde{K}_Q((z - x)/l(Q))\right|$ for points $z \in S_j$ and $x \in Q$ this becomes

$$\begin{aligned} \left\|D^\alpha \mathcal{E}_Q f\right\|^p_{L^p(Q)} &\le C \frac{1}{l(Q)^n} \int_Q \sum_j \left(\frac{l(Q)}{l(S_j)}\right)^{n-1} \kappa\left(\frac{l(S_j)}{l(Q)}\right) \int_{S_j} \left|D^\alpha f(z)\right|^p dz\, dx \\ &\le C \sum_j \left(\frac{l(Q)}{l(S_j)}\right)^{n-1} \kappa\left(\frac{l(S_j)}{l(Q)}\right) \int_{S_j} \left|D^\alpha f(z)\right|^p dz \end{aligned}$$

because the integrand is then independent of $x \in Q$.

It is now possible to sum over all $Q \in \mathcal{W}_3$. Let $\mathcal{G}(S)$ be the set of all cubes $Q \in \mathcal{W}_3$ such that the twisting cone corresponding to Q intersects the Whitney cube S of Ω. and recall (27) in which we bounded the number of cubes of size $l(Q) = 2^{-m} l(S)$ in

$\mathcal{G}(S)$ by $C(n,\epsilon)2^{nm}$. This yields

$$\begin{aligned}
\sum_{Q\in\mathcal{W}_2} \left\|D^\alpha \mathcal{E}_Q f\right\|^p_{L^p(Q)} &\le C \sum_{Q\in\mathcal{W}_3} \sum_{S_j\cap\Gamma_Q} \left(\frac{l(Q)}{l(S_j)}\right)^{n-1} \kappa\left(\frac{l(S_j)}{l(Q)}\right) \int_{S_j} \left|D^\alpha f(z)\right|^p dz \\
&\le C \sum_{S\in\mathcal{W}_1} \left\|D^\alpha f(z)\right\|^p_{L^p(S)} \sum_{Q\in\mathcal{G}(S)} \left(\frac{l(Q)}{l(S)}\right)^{n-1} \kappa\left(\frac{l(S)}{l(Q)}\right) \\
&\le C \sum_{S\in\mathcal{W}_1} \left\|D^\alpha f(z)\right\|^p_{L^p(S)} \left(\sum_m 2^{nm}2^{-m(n-1)}\kappa(2^m)\right) \\
&\le C \sum_{S\in\mathcal{W}_1} \left\|D^\alpha f(z)\right\|^p_{L^p(S)} \\
&= C\left\|D^\alpha f(z)\right\|^p_{L^p(\Omega)}
\end{aligned}$$

where in the penultimate step we used the bound from Lemma 27.

Estimates for Adjacent Cubes

Our goal is an estimate needed to prove compatibility of the extensions for pairs of adjacent cubes.

Lemma 30 *Let $\mathcal{N}(Q')$ be the collection of cubes from $\mathcal{W}_2$ that are adjacent to Q'. If α is a multi-index with $|\alpha| \le k$ then for $1 \le p < \infty$*

$$\begin{aligned}
\sum_{Q'\in\mathcal{W}_2} \sum_{Q\in\mathcal{N}(Q')} \sum_{0\le\beta\le\alpha} c(|\alpha-\beta|)^p l(Q')^{-|\alpha-\beta|p} \left\|D^\beta(\mathcal{E}_Q f - \mathcal{E}_{Q'} f)\right\|^p_{L^p(Q'\cap(17/16)Q)} \\
\le C(n,\epsilon,\delta,k,p)\left\|\nabla^k f(y)\right\|^p_{L^p(\Omega)} \qquad (82)
\end{aligned}$$

while for $p=\infty$ we have for $x\in Q'$

$$l(Q')^{-|\alpha-\beta|}\left|D^\beta(\mathcal{E}_Q f(x) - \mathcal{E}_{Q'} f(x))\right| \le C\,\|\nabla^k f\|_{L^\infty(\Omega)} \qquad (83)$$

PROOF. If either Q or Q' is in $\mathcal{W}_2 \setminus \mathcal{W}_3$ then their adjacency ensures that both have length at least $\epsilon\delta/(50n)$. In that case (8) shows that all cubes in the chains covering Γ_Q and $\Gamma_{Q'}$ have length at least $2\epsilon\delta/(25\sqrt{n})$. Our assumption on the support of f then guarantees $f \equiv 0$ on the twisting cones, whence $\mathcal{E}_Q \equiv 0 \equiv \mathcal{E}_{Q'}$. No estimate is needed here, so we henceforth assume both Q and Q' are in $\mathcal{W}_3$.

Recall that the twisting cone Γ_Q corresponding to Q has a central curve γ_Q and at each $z\in\gamma_Q$ a radius $s(z)$. The initial point of γ is called z_0 and the ball B_0 is $B_0 = B(z_0, s(z_0))$. Analogous definitions are made for γ', z_0', and B_0'. Before Lemma 15 we defined the polynomial fitted to a function on a set; here we let P_Q be the degree $(k-1)$ polynomial fitted to f on B_0 and $P_{Q'}$ be the corresponding polynomial for f on B_0'. It will be convenient to denote convolution with the scaling parameter $l(Q)$ by

$$g * \tilde{K}_Q(x) = \int_{\mathbb{R}^n} g(x + l(Q)\tilde{y})\tilde{K}_Q(\tilde{y})\, d\tilde{y} \tag{84}$$

and to express the difference to be estimated as

$$\mathcal{E}_Q f(x) - \mathcal{E}_{Q'} f(x) = ((f_Q - P_Q) * \tilde{K}_Q) + (P_Q * \tilde{K}_Q) - (P_{Q'} * \tilde{K}_{Q'}) - ((f_{Q'} - P_{Q'}) * \tilde{K}_{Q'}). \tag{85}$$

If $1 \le p < \infty$ we take the derivative D^β, the p-th power, and the integral over $(Q' \cap (17/16)Q)$. Using the fact that there are only three terms in the sum we have

$$\begin{aligned}
&\left\| D^\beta(\mathcal{E}_Q f - \mathcal{E}_{Q'} f) \right\|^p_{L^p(Q' \cap (17/16)Q)} \\
&\le C(p) \left\| D^\beta((f_Q - P_Q) * \tilde{K}_Q) \right\|^p_{L^p((17/16)Q)} + C(p) \left\| D^\beta((f_{Q'} - P_{Q'}) * \tilde{K}_{Q'}) \right\|^p_{L^p(Q')} \\
&\quad + C(p) \left\| D^\beta(P_Q * \tilde{K}_Q - P_{Q'} * \tilde{K}_{Q'}) \right\|^p_{L^p(Q')}
\end{aligned}$$

The two types of terms in this expression are individually estimated in Lemmas 31 and 32; substituting from these completes the proof in the case $1 \le p < \infty$.

When $p = \infty$ we directly apply (85) and the L^∞ estimates of Lemmas 31 and 32. The result has an additional $l(Q')^{k-|\alpha|}$ factor, but this is bounded because $|\alpha| \le k$ and the cubes are from $\mathcal{W}_3$.

Polynomial Terms

Lemma 31 *There are constants $C = C(n, \epsilon, \delta, k, p)$ such that for $1 \le p < \infty$*

$$\sum_{Q' \in \mathcal{W}_1} \sum_{Q \in \mathcal{N}(Q')} \sum_{0 \le \beta \le \alpha} l(Q')^{-|\alpha-\beta|p} \left\| D^\beta(P_Q * \tilde{K}_Q - P_{Q'} * \tilde{K}_{Q'}) \right\|^p_{L^p(Q')} \le C \left\| \nabla^k f(y) \right\|^p_{L^p(\Omega)}$$

while for $p = \infty$

$$l(Q')^{-|\alpha-\beta|} \left\| D^\beta(P_Q * \tilde{K}_Q - P_{Q'} * \tilde{K}_{Q'}) \right\|_{L^\infty(Q')} \le C \left\| \nabla^k f(y) \right\|_{L^\infty(\Omega)} l(Q')^{k-|\alpha|}.$$

PROOF. Expanding the polynomial $P_Q(x + l(Q)\tilde{y})$ as a polynomial in $l(Q)\tilde{y}$ and using the property (74) of the kernel $\tilde{K}_Q$ we see

$$P_Q * \tilde{K}_Q(x) = \int_{\mathbb{R}^n} P_Q(x + l(Q)\tilde{y})\tilde{K}_Q(\tilde{y})\, d\tilde{y} = P_Q(x) \tag{86}$$

Similarly $P_{Q'} * \tilde{K}_{Q'}(x) = P_{Q'}(x)$ and it suffices to estimate terms $\left\| D^\beta(P_Q - P_{Q'}) \right\|_{L^p(Q')}$. At this point we could appeal to Lemma 3.2 of [11] in which precisely this is proved, but for the convenience of the reader we instead sketch a proof using Lemma 15.

From (8) and (9) we see that the diameter of B_0' is comparable both to $l(Q')$ and to $\operatorname{dist}(Q', B_0')$. This ensures that the finite dimensional Banach spaces $L_k^p(Q')$ and $L_k^p(B_0')$ have equivalent norms, so we may write

$$
\begin{aligned}
\left\|D^\beta(P_Q - P_{Q'})\right\|_{L^p(Q')} &\leq C\left\|D^\beta(P_Q - P_{Q'})\right\|_{L^p(B_0')} \\
&\leq \left\|D^\beta(f - P_{Q'})\right\|_{L^p(B_0')} + \left\|D^\beta(f - P_Q)\right\|_{L^p(B_0')} \\
&\leq Cs'(z_0')^{k-|\beta|}\|\nabla^k f\|_{L^p(B_0')} + \left\|D^\beta(f - P_Q)\right\|_{L^p(B_0')} \qquad (87)
\end{aligned}
$$

where we have used the Poincaré inequality (12) on B_0'. Now $D^\beta P_Q$ is precisely the polynomial fitted to $D^\beta f$ on B_0. Let $\{T_j\}$ be the chain of cubes connecting the centers of the balls B_0 and B_0'. By Lemma 10 we have a bound on the number of cubes in the chain and know that all of them satisfy $C^{-1} \leq l(T_j)/l(Q') \leq C$. Restricting to the case $1 \leq p < \infty$ and applying Lemma 15 yields

$$
\begin{aligned}
\left\|D^\beta(f - P_Q)\right\|_{L^p(B_0')} &\leq C\,(l(T_m))^{k-|\beta|-1} \sum_{j=1}^{m} l(T_j)\left(\frac{l(T_m)}{l(T_j)}\right)^{n/p} \left\|\nabla^k f(y)\right\|_{L^p(T_j)} \\
&\leq C\,l(Q')^{k-|\beta|} \sum_{j=1}^{m}\left\|\nabla^k f(y)\right\|_{L^p(T_j)}
\end{aligned}
$$

After combining this with (87) we may use Hölder's inequality and the bound on the number of cubes in the chain to estimate the p-th power by

$$
\begin{aligned}
\left\|D^\beta(P_Q - P_{Q'})\right\|_{L^p(B_0')}^p &\leq Cs'(z_0')^{(k-|\beta|)p}\|\nabla^k f\|_{L^p(B_0')}^p + C\,l(Q')^{(k-|\beta|)p} \sum_{j=1}^{m}\left\|\nabla^k f(y)\right\|_{L^p(T_j)}^p \\
&\leq C\,l(Q')^{(k-|\beta|)p} \sum_{j=1}^{m}\left\|\nabla^k f(y)\right\|_{L^p(T_j)}^p \qquad (88)
\end{aligned}
$$

where we have also used $s'(z_0') \leq Cl(Q')$.

To perform the summation in the statement of the lemma we need the estimate (26). It is apparent that for appropriate choices of the constants in (25) our chain $\{T_l\}$ joins cubes S and S' from $\mathcal{F}(Q')$, whereupon we may calculate from (88) and

(26)

$$
\begin{aligned}
&\sum_{Q'\in\mathcal{W}_3}\sum_{Q\in\mathcal{N}(Q)}\sum_{0\le\beta\le\alpha} l(Q')^{-|\alpha-\beta|p}\left\|P_Q * \tilde{K}_Q - P_{Q'} * \tilde{K}_{Q'}\right\|^p_{L^p(Q')}\\
&\le C\sum_{Q'\in\mathcal{W}_3}\sum_{0\le\beta\le\alpha} l(Q')^{-|\alpha-\beta|p} l(Q')^{(k-|\beta|)p}\sum_{S,S'\in\mathcal{F}(Q')}\sum_{T_l(S,S')}\|\nabla^k f\|^p_{L^p(T)}\\
&\le C\sum_{Q'\in\mathcal{W}_3}\sum_{S,S'\in\mathcal{F}(Q')}\sum_{T_l(S,S')}\|\nabla^k f\|^p_{L^p(T)} l(Q')^{(k-|\alpha|)p}\\
&\le C\sum_{Q'\in\mathcal{W}_3}\sum_{S,S'\in\mathcal{F}(Q')}\sum_{T_l(S,S')}\|\nabla^k f\|^p_{L^p(T)}\\
&\le C\sum_{T\in\mathcal{W}_1}\|\nabla^k f\|^p_{L^p(T)} = C\,\|\nabla^k f\|^p_{L^p(\Omega)}
\end{aligned}
$$

Observe that in the third to last step we used that $|\alpha| \le k$ and that there is a bound on the size of cubes $Q' \in \mathcal{W}_3$. It is easy to verify that all constants introduced depend only upon n, ϵ, δ, k and p, so this concludes the proof for the case $1 \le p < \infty$.

To complete the proof for $f \in L^\infty_k(\Omega)$ we use (15) of Lemma 15 to write

$$
\begin{aligned}
\left\|D^\beta(P_Q - P_{Q'})\right\|_{L^\infty(B_0')} &\le \left\|D^\beta(f - P_{Q'})\right\|_{L^\infty(B_0')} + \left\|D^\beta(f - P_Q)\right\|_{L^\infty(B_0')}\\
&\le Cs'(z_0')^{k-|\beta|}\|\nabla^k f\|_{L^\infty(B_0')} + C\,l(Q')^{k-|\beta|}\|\nabla^k f\|_{L^\infty(\Omega)}\\
&\le\ l(Q')^{k-|\beta|}\|\nabla^k f\|_{L^\infty(\Omega)}
\end{aligned}
$$

because both the diameter of B_0' and the separation of B_0 from B_0' are comparable to $l(Q')$. Substituting into (87) and multiplying by $l(Q')^{-|\alpha-\beta|}$ then gives the result.

Terms involving $(\mathbf{f} - \mathbf{P_Q})$

Lemma 32 *There are constants $C = C(n, \epsilon, \delta, k, p)$ such that for $1 \le p < \infty$*

$$
\sum_{Q'\in\mathcal{W}_3}\sum_{Q\in\mathcal{N}(Q')}\sum_{0\le\beta\le\alpha} l(Q)^{-|\alpha-\beta|p}\left\|D^\beta((f_Q - P_Q)*\tilde{K}_Q)\right\|^p_{L^p((17/16)Q)} \le C\left\|\nabla^k f(y)\right\|^p_{L^p(\Omega)} \tag{89}
$$

while for $p = \infty$

$$
l(Q)^{-|\alpha-\beta|}\left\|D^\beta((f_Q - P_Q) * \tilde{K}_Q)\right\|_{L^\infty((17/16)Q)} \le C\left\|\nabla^k f(y)\right\|_{L^\infty(\Omega)} l(Q)^{k-|\alpha|}. \tag{90}
$$

PROOF. We first differentiate within the integral (77) and make the change of variables $z = (x + l(Q)\tilde{y})$ to obtain

$$
\begin{aligned}
D^\beta((f_Q - P_Q) * \tilde{K}_Q)(x) &= \int_{\mathbb{R}^n} D^\beta(f_Q - P_Q)(x + l(Q)\tilde{y})\tilde{K}_Q(\tilde{y})\,d\tilde{y} \\
&= \frac{1}{l(Q)^n}\int_{\mathbb{R}^n} D^\beta(f_Q - P_Q)(z)\tilde{K}_Q\left(\frac{z-x}{l(Q)}\right)dz
\end{aligned}
$$

By Lemma 26 we know that all points at which $\tilde{K}_Q((z - x)/l(Q)) \neq 0$ lie either in the union of cubes S_j from the chain covering Γ_Q, or within distance $\sqrt{n}l(Q)$ of $\Gamma_Q^* \setminus \Gamma_Q$. It is possible from the definition of Γ_Q^* to define a collection $\{T_m\}$ of cubes such that each T_m has length comparable to its separation from Q and so $\cup T_m$ contains all points within distance $\sqrt{n}l(Q)$ of $\Gamma_Q^* \setminus \Gamma_Q$. All of the constants of comparability depend on n, ϵ, and δ and in particular it is evident that (75) is still valid for these new cubes. We may then adjoin $\{T_m\}$ to the chain $\{S_j\}$ so that we have a chain covering all of Γ_Q^*. Abusing notation we also call the new chain $\{S_j\}$. Not all cubes in this chain are Whitney cubes of Ω, but in our working we need only keep in mind that $f_Q \equiv 0$ on all those that are not. Using this convention, (75) implies

$$
\left|D^\beta((f_Q - P_Q) * \tilde{K}_Q)(x)\right| \leq \sum_j \left(\frac{l(Q)}{l(S_j)}\right)^{n-1} \kappa\left(\frac{l(S_j)}{l(Q)}\right)\int_{S_j}\left|D^\beta(f_Q - P_Q)(z)\right|\frac{dz}{l(Q)^n}. \quad (91)
$$

Now suppose $1 \leq p < \infty$ and apply (14) of Lemma 15 with the exponent $p = 1$ to the integrals. This gives

$$
\begin{aligned}
\int_{S_j}\left|D^\beta(f_Q - P_Q)(z)\right|\frac{dz}{l(Q)^n} &\leq \frac{C}{l(Q)^n}(l(S_j))^{k-|\beta|-1}\sum_{m=1}^{j} l(S_m)\left(\frac{l(S_j)}{l(S_m)}\right)^n \left\|\nabla^k f(y)\right\|_{L^1(S_m)} \\
&= C\,(l(S_j))^{k-|\beta|-1}\left(\frac{l(S_j)}{l(Q)}\right)^n \sum_{m=1}^{j} l(S_m) \fint_{S_m}\left|\nabla^k f_Q(y)\right|dy
\end{aligned}
$$

This is even valid on the cubes that we appended to the chain, bearing in mind that $f_Q \equiv 0$ on those cubes. Substituting back into (91)

$$
\begin{aligned}
&\left|D^\beta((f_Q - P_Q) * \tilde{K}_Q)(x)\right| \\
&\qquad \leq C\,\sum_j \left(\frac{l(S_j)}{l(Q)}\right)\kappa\left(\frac{l(S_j)}{l(Q)}\right)(l(S_j))^{k-|\beta|-1}\sum_{m=1}^{j} l(S_m)\fint_{S_m}\left|\nabla^k f_Q(y)\right|dy \\
&\qquad = C\,l(Q)^{k-|\beta|}\sum_j \left(\frac{l(S_j)}{l(Q)}\right)^{k-|\beta|}\kappa\left(\frac{l(S_j)}{l(Q)}\right)\sum_{m=1}^{j}\frac{l(S_m)}{l(Q)}\fint_{S_m}\left|\nabla^k f_Q(y)\right|dy \\
&\qquad = C\,l(Q)^{k-|\beta|}\sum_m \frac{l(S_m)}{l(Q)}\fint_{S_m}\left|\nabla^k f_Q(y)\right|dy\left[\sum_{j=m}^{\infty}\left(\frac{l(S_j)}{l(Q)}\right)^{k-|\beta|}\kappa\left(\frac{l(S_j)}{l(Q)}\right)\right]
\end{aligned}
$$

however the number of S_j of a given scale is bounded by constants depending on n, ϵ and δ, so applying Lemma 27

$$\sum_{j=m}^{\infty}\left(\frac{l(S_j)}{l(Q)}\right)^{k-|\beta|}\kappa\left(\frac{l(S_j)}{l(Q)}\right)\le C(n,\epsilon,\delta,k)\left(\frac{l(S_m)}{l(Q)}\right)^{k-|\beta|}\kappa\left(\frac{l(S_m)}{l(Q)}\right)$$

and hence

$$\left|D^\beta((f_Q-P_Q)*\tilde{K}_Q)(x)\right|\le C\,l(Q)^{k-|\beta|}\sum_m\left(\frac{l(S_m)}{l(Q)}\right)^{k-|\beta|+1}\kappa\left(\frac{l(S_m)}{l(Q)}\right)\fint_{S_m}\left|\nabla^k f_Q(y)\right|dy$$

Taking the p-th power we may use Hölder's inequality, then the estimate from Lemma 27 with $q=(kp-|\beta|p+p-n)/(p-1)$, and then Jensen's inequality to conclude

$$\begin{aligned}&\left|D^\beta((f_Q-P_Q)*\tilde{K}_Q)(x)\right|^p\\&\le C\,l(Q)^{(k-|\beta|)p}\left[\sum_{m=1}^{\infty}\left(\frac{l(S_m)}{l(Q)}\right)^n\kappa\left(\frac{l(S_m)}{l(Q)}\right)\left(\fint_{S_m}|\nabla^k f_Q(y)|\,dy\right)^p\right]\left[\sum_{m=1}^{\infty}\left(\frac{l(S_m)}{l(Q)}\right)^q\kappa\left(\frac{l(S_m)}{l(Q)}\right)\right]^{p-1}\\&\le C\,l(Q)^{(k-|\beta|)p}\sum_{m=1}^{\infty}\left(\frac{l(S_m)}{l(Q)}\right)^n\kappa\left(\frac{l(S_m)}{l(Q)}\right)\fint_{S_m}|\nabla^k f_Q(y)|^p\,dy\\&\le Cl(Q)^{(k-|\beta|)p-n}\sum_{m=1}^{\infty}\kappa\left(\frac{l(S_m)}{l(Q)}\right)\int_{S_m}|\nabla^k f_Q(y)|^p\,dy\end{aligned}$$

As the estimate is independent of x, integration over $(17/16)Q$ merely increases the constant marginally and cancels a factor of $l(Q)^{-n}$. Thus

$$\left\|D^\beta((f_Q-P_Q)*\tilde{K}_Q)(x)\right\|^p_{L^p((17/16)Q)}\le Cl(Q)^{(k-|\beta|)p}\sum_{m=1}^{\infty}\kappa\left(\frac{l(S_m)}{l(Q)}\right)\int_{S_m}|\nabla^k f_Q(y)|^p\,dy. \tag{92}$$

If we multiply (92) by $l(Q)^{-|\alpha-\beta|p}$ and sum as in (89) we obtain

$$\begin{aligned}&\sum_{Q'\in\mathcal{W}_3}\sum_{Q\in\mathcal{N}(Q')}\sum_{0\le\beta\le\alpha}l(Q)^{-|\alpha-\beta|p}\left\|D^\beta((f_Q-P_Q)*\tilde{K}_Q)\right\|^p_{L^p((17/16)Q)}\\&\qquad\le C\sum_{Q'\in\mathcal{W}_3}\sum_{Q\in\mathcal{N}(Q')}\sum_{0\le\beta\le\alpha}l(Q)^{(k-|\alpha|)p}\sum_{S_m\cap\Gamma^*_Q\neq\emptyset}\kappa\left(\frac{l(S_m)}{l(Q)}\right)\int_{S_m}|\nabla^k f_Q(y)|^p\,dy\end{aligned}$$

but we have bounds for the number of neighbors $Q\in\mathcal{N}(Q')$ and the values β with $0\le\beta\le\alpha$ and $|\alpha|\le k$. Moreover $Q\in\mathcal{W}_3$ has $l(Q)^{(k-|\alpha|)p}\le 1$ for $|\alpha|\le k$. If we write $\mathcal{W}_4$ for the collection of cubes that are neighbors of cubes from $\mathcal{W}_3$ the estimate then reduces to

$$\sum_{Q'\in\mathcal{W}_3}\sum_{Q\in\mathcal{N}(Q')}\sum_{0\le\beta\le\alpha} l(Q)^{-|\alpha-\beta|p}\left\|D^\beta((f_Q-P_Q)*\tilde{K}_Q)\right\|^p_{L^p((17/16)Q)}$$
$$\le C\sum_{Q\in\mathcal{W}_4}\sum_{S_m\cap\Gamma^*_Q\neq\emptyset}\kappa\left(\frac{l(S_m)}{l(Q)}\right)\int_{S_m}|\nabla^k f_Q(y)|^p\,dy$$

Note that since $f_Q\equiv 0$ on the cubes S_j that do not intersect Γ_Q we may leave those out of the inner sum. The cubes that remain are Whitney cubes of Ω on which $f_Q\equiv f$. Reversing the order of summation we find that for each $S\in\mathcal{W}_1$ we sum over $Q\in\mathcal{G}(S)$, where $\mathcal{G}(\mathcal{S})$ is as in (27). It was proven in (27) that the number of these cubes having scale $2^{-j}l(S)$ is bounded by a constant multiple of 2^{nj}, so

$$\sum_{Q'\in\mathcal{W}_3}\sum_{Q\in\mathcal{N}(Q')}\sum_{0\le\beta\le\alpha} l(Q)^{-|\alpha-\beta|p}\left\|D^\beta((f_Q-P_Q)*\tilde{K}_Q)\right\|^p_{L^p((17/16)Q)}$$
$$\le C\sum_{Q\in\mathcal{W}_4}\sum_{S_m\cap\Gamma_Q\neq\emptyset}\kappa\left(\frac{l(S_m)}{l(Q)}\right)\int_{S_m}|\nabla^k f(y)|^p\,dy$$
$$= C\sum_{S\in\mathcal{W}_1}\int_S|\nabla^k f(y)|^p\,dy\sum_{Q\in\mathcal{G}(S)}\kappa\left(\frac{l(S_m)}{l(Q)}\right)$$
$$\le C\sum_{S\in\mathcal{W}_1}\int_S|\nabla^k f(y)|^p\,dy\sum_{j=0}^{\infty}2^{nj}k(2^j)$$
$$\le C\sum_{S\in\mathcal{W}_1}\int_S|\nabla^k f(y)|^p\,dy$$
$$\le C\left\|\nabla^k f(y)\right\|^p_{L^p(\Omega)}$$

where the penultimate estimate is from Lemma 27.

As has been true throughout, the proof is easier in the case $p=\infty$. Returning to (91) we need only use (15) of Lemma 15 to deduce

$$\left|D^\beta((f_Q-P_Q)*\tilde{K}_Q)(x)\right|\le\|\nabla^k f\|_{L^\infty(\Omega)}\sum_j\left(\frac{l(S_j)}{l(Q)}\right)l(S_j)^{k-|\beta|}\kappa\left(\frac{l(S_j)}{l(Q)}\right)$$
$$\le C\,l(Q)^{k-|\beta|}\|\nabla^k f\|_{L^\infty(\Omega)}\sum_j\left(\frac{l(S_j)}{l(Q)}\right)^{k-|\beta|+1}\kappa\left(\frac{l(S_j)}{l(Q)}\right)$$
$$\le C\,l(Q)^{k-|\beta|}\|\nabla^k f\|_{L^\infty(\Omega)}$$

where we used the fact that only finitely many S_j of a given scale intersect the twisting cone, and the estimate from Lemma 27. Multiplying by $l(Q)^{-|\alpha-\beta|}$ gives the desired result.

5 Proof of Theorem 8

Definition and Bounds for the Extension

Using standard techniques we may construct a smooth partition of unity corresponding to the Whitney decomposition $\mathcal{W}_2$. In particular, from Stein [18] Chapter VI Section 1.3 there are C^∞ functions Φ_Q such that $\sum \Phi_Q \equiv 1$ on the interior of Ω^c, there are bounds $0 \le \Phi_Q \le 1$, the support of each Φ_Q is in $(17/16)Q$, and the derivatives satisfy $|D^\alpha \Phi_Q| \le c(|\alpha|) l(Q)^{-|\alpha|}$. Fix such a partition and define for $f \in L^p_k(\Omega)$

$$\mathcal{E}f(x) = \begin{cases} f(x) & \text{if } x \in \Omega \\ \sum_{Q \in \mathcal{W}_2} \Phi_Q(x) \mathcal{E}_Q f(x) & \text{if } x \in (\Omega^c)^o \end{cases}$$

The definition of locally uniform implies that $\partial\Omega$ has no density points and is therefore of measure zero, so $\mathcal{E}f$ is defined almost everywhere. Moreover the properties we have established for the $\mathcal{E}_Q$ allow us to bound the Sobolev norm of this function on $(\Omega^c)^o$. We begin by computing

$$\begin{aligned} D^\alpha \mathcal{E}f &= D^\alpha (\mathcal{E}_{Q'} f + \sum_{Q \in \mathcal{W}_2} (\mathcal{E}_Q f - \mathcal{E}_{Q'} f) \Phi_Q) \\ &= D^\alpha \mathcal{E}_{Q'} f + \sum_{Q \in \mathcal{W}_2} \sum_{0 \le \beta \le \alpha} D^\beta (\mathcal{E}_Q f - \mathcal{E}_{Q'} f)\, D^{\alpha-\beta} \Phi_Q. \end{aligned} \tag{93}$$

Using the notation $\mathcal{N}(Q')$ for the cubes neighboring Q' and inserting the bound on the derivatives of the partition of unity we obtain for $1 \le p < \infty$

$$\begin{aligned} &\|D^\alpha \mathcal{E} f\|^p_{L^p(Q')} \\ &\quad \le \Big(\|D^\alpha \mathcal{E}_{Q'} f\|_{L^p(Q')} \\ &\qquad + \sum_{Q \in \mathcal{N}(Q')} \sum_{0 \le \beta \le \alpha} c(|\alpha - \beta|) l(Q')^{-|\alpha-\beta|} \|D^\beta (\mathcal{E}_Q f - \mathcal{E}_{Q'} f)\|_{L^p(Q' \cap (17/16)Q)} \Big)^p \\ &\quad \le C(n,k,p)\, \|D^\alpha \mathcal{E}_{Q'} f\|^p_{L^p(Q')} \\ &\qquad + C(n,k,p) \sum_{Q \in \mathcal{N}(Q')} \sum_{0 \le \beta \le \alpha} c(|\alpha - \beta|)^p l(Q')^{-|\alpha-\beta|p} \|D^\beta (\mathcal{E}_Q f - \mathcal{E}_{Q'} f)\|^p_{L^p(Q' \cap (17/16)Q)} \end{aligned}$$

where the latter inequality uses the Hölder estimate and the fact that the number of terms in the sum depends only on n and k. If we then sum over all $Q' \in \mathcal{W}_2$ and use the bounds (79) and (82) we find

$$
\begin{aligned}
&\left\|D^\alpha \mathcal{E} f\right\|^p_{L^p\left((\Omega^c)^o\right)} \\
&\quad \le C \sum_{Q' \in \mathcal{W}_2} \|D^\alpha \mathcal{E}_{Q'} f\|^p_{L^p(Q')} \\
&\qquad + \sum_{Q' \in \mathcal{W}_2} \sum_{Q \in \mathcal{N}(Q')} \sum_{0 \le \beta \le \alpha} c(|\alpha-\beta|) l(Q')^{-|\alpha-\beta|} \|D^\beta(\mathcal{E}_Q f - \mathcal{E}_{Q'} f)\|_{L^p(Q' \cap (17/16)Q)} \\
&\quad \le C \left\|D^\alpha f(z)\right\|^p_{L^p(\Omega)} + C \left\|\nabla^k f(y)\right\|^p_{L^p(\Omega)}
\end{aligned}
$$

with constants that now depend on n, k, p, ϵ and δ. Summing over $|\alpha| \le k$ bounds the $L^p_k((\Omega^c)^o)$ norm of $\mathcal{E}f$ by the $L^p_k(\Omega)$ norm of f. A similar bound is valid in the $p = \infty$ case, where we instead take absolute values and the bounds on derivatives of Φ_Q into (93), use (80), (83), and the fact that the summation over multi-indices and neighboring cubes only introduces a constant factor, to obtain

$$
\begin{aligned}
|D^\alpha \mathcal{E} f| &\le |D^\alpha \mathcal{E}_{Q'} f| + \sum_{Q \in \mathcal{N}(Q')} \sum_{0 \le \beta \le \alpha} c(|\alpha-\beta|) l(Q')^{-|\alpha-\beta|} |D^\beta(\mathcal{E}_Q f - \mathcal{E}_{Q'} f)| \\
&\le C \|D^\alpha f\|_{L^\infty(\Omega)} + C \left\|\nabla^k f(y)\right\|_{L^\infty(\Omega)} \qquad (94)
\end{aligned}
$$

and then sum over $|\alpha| \le k$.

What remains to be proven is that $\mathcal{E}f$ is in $L^p_k(\mathbb{R}^n)$. This may be thought of as checking that the pieces of $\mathcal{E}f$ "join up" correctly at $\partial\Omega$, and is not too difficult to verify in the case that $f \in C^\infty$ with bounded derivatives. We reduce to this case using the following result of Jones (Proposition 4.4 in [11]).

Proposition 33 (Jones) *For fixed $\eta > 0$, k, $p \in [1, \infty)$, and $f \in L^p_k(\Omega)$ there is $g \in C^\infty(\mathbb{R}^n) \cap L^p_k(\Omega)$ and $M \in \mathbb{R}$ with*

$$
\|f - g\|_{L^p_k(\Omega)} \le C\eta \qquad \text{and} \qquad |D^\alpha g| \le M \quad \text{for } 0 \le |\alpha| \le k \qquad (95)
$$

while for fixed $f \in L^\infty_k(\Omega)$ there is $g \in C^\infty(\mathbb{R}^n) \cap L^\infty_k(\Omega)$ with

$$
\|f - g\|_{L^\infty_{k-1}(\Omega)} \le C\eta \qquad \text{and} \qquad \|g\|_{L^\infty_k(\Omega)} \le C\|f\|_{L^\infty_k(\Omega)} \qquad (96)
$$

Fix α with $|\alpha| < k$. For f satisfying the conditions of Proposition 33 we see from (95) or (96) that $D^\alpha \mathcal{E} f$ is Lipschitz in a neighborhood of any point of Ω, and by virtue of the L^∞ estimate (94) it is also Lipschitz in a neighborhood of any point of $(\Omega^c)^o$. We claim that this still holds in a neighborhood of any point of $\partial\Omega$, and therefore that $D^\alpha \mathcal{E} f$ is locally Lipschitz. It clearly suffices that there is a constant $s > 0$ such that if $x \in (\Omega^c)^o$ and $y \in \Omega$ with $|x - y| < s$ then

$$
\left|D^\alpha(\mathcal{E} f(x) - \mathcal{E} f(y))\right| \le C|x - y|. \qquad (97)
$$

Let $s = \epsilon\delta/200n$, fix $x \in (\Omega^c)^o$ and $y \in \Omega$ with $|x - y| < s$. Let $Q \in \mathcal{W}_3$ contain x

and x_Q be its center, and set y_Q to be the initial point of the curve γ around which we have the twisting cone Γ_Q. Integration against $\tilde{K}_Q$ preserves polynomials, so in particular it preserves the constant $L = D^\alpha f(y_Q)$. Since $\mathcal{E} f(x_Q) = \mathcal{E}_Q f(x_Q)$ we may compute

$$\begin{aligned}\left|D^\alpha \mathcal{E} f(x_Q) - D^\alpha f(y_Q)\right| &= \left|\int_{\mathbb{R}^n} (D^\alpha f_Q(x_Q + l(Q)\tilde{y}) - L)\tilde{K}_Q(\tilde{y})\, d\tilde{y}\right| \\ &\leq \int_{\mathbb{R}^n} \left|D^\alpha f_Q(x_Q + l(Q)\tilde{y}) - L\right|\left|\tilde{K}_Q(\tilde{y})\right| d\tilde{y}\end{aligned}$$

Reasoning as in the proof of the L^∞ estimate for Lemma 32 we see that

$$\left|D^\alpha f_Q(x{+}l(Q)\tilde{y}){-}L\right| = \left|D^\alpha f_Q(x_Q{+}l(Q)\tilde{y}){-}D^\alpha f(y_Q)\right| \leq C\left|x_Q{+}l(Q)\tilde{y}{-}y_Q\right|^{k-|\alpha|} ||\nabla^k f||_{L^\infty(\Omega)}$$

and this may be integrated against $|\tilde{K}_Q|$ to provide

$$\left|D^\alpha \mathcal{E} f(x_Q) - D^\alpha f(y_Q)\right| \leq C\, l(Q)^{k-|\alpha|} ||\nabla^k f||_{L^\infty(\Omega)} \leq C\, |x-y|^{k-|\alpha|} ||\nabla^k f||_{L^\infty(\Omega)} \qquad (98)$$

From Lemma 12 we know $|x - x_Q| \leq \mathrm{dist}(x_Q, \Omega) \leq |x - y|$, and combining this with our bound on $|D^\alpha \mathcal{E} f|_{L^\infty((\Omega^c)^o)}$ shows that

$$\left|\mathcal{E} f(x) - \mathcal{E} f(x_Q)\right| \leq C ||\nabla^k f||_{L^\infty(\Omega)} |x-y|. \qquad (99)$$

Also from this lemma we have $|x_Q - y_Q| \leq 20\sqrt{n} l(Q) \leq C|x-y|$, so $|y_Q - y| \leq 25\sqrt{n}|x-y| < \delta$. We may therefore connect y to y_Q with a chain of cubes and apply the L^∞ estimate in Lemma 15 to conclude

$$\left|D^\alpha f(y) - D^\alpha f(y_Q)\right| \leq C ||\nabla^k f||_{L^\infty(\Omega)} |x-y|^{k-|\alpha|}$$

This may be combined with (98), (99), and the fact $|x - y| < 1$ to prove (97).

The above reasoning shows that any f satisfying the conclusions of Proposition 33 has locally Lipschitz derivatives of all orders less than k and is therefore k-times differentiable almost everywhere. We conclude that $f \in L^p_k(\mathbb{R}^n)$ and

$$||\mathcal{E} f||_{L^p_k(\mathbb{R}^n)} \leq C ||f||_{L^p_k(\Omega)}$$

so that $\mathcal{E}$ is a bounded linear operator on this space of functions. Proposition 33 shows that we can approximate (or weakly approximate in the case $p = \infty$) any $g \in L^p_k(\Omega)$ by such f, and consequently that $\mathcal{E} g$ is in $L^p_k(\mathbb{R}^n)$ and satisfies the same estimate. This completes the proof of Theorem 8.

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